Helping Students Become Effective Mathematical Problem Solvers

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A Discussion in 3 Parts

1. Framing the discussion: What is "problem solving," and what are reasonable goals for it?
2. What are the attributes of good problems, and instruction that makes the most of them?
3. What are the attributes of "powerful instruction" – instruction that produces students who are powerful thinkers and problem solvers?

Part 1:
Framing the discussion

What is Problem Solving?

A Working Definition:

You are engaged in Problem Solving when you are trying to achieve something, and you do not know a straightforward way to do so.

Examples:

Finding the product of two 37-digit numbers is NOT problem solving. (It's hard and you may goof, but you know how to do it.)

Writing an essay trying to convince someone of your perspective; and

Working a mathematics problem where you have to make sense of it and figure out what to do,

ARE acts of problem solving.

What does it mean to be a good problem solver?

The individual has to:

- be willing to dig into new problems,
- have some relevant knowledge,
- be a flexible thinker, and
- be willing to persevere in the face of difficulty.
- In fact, the research says that the following 4 things determine the success or failure of problem solving attempts …

The Big Picture

These four categories of knowledge determine the quality (and success) of problem solving attempts:

A. The knowledge base
B. Problem solving strategies (heuristics)
C. "Control": monitoring and self-regulation, or metacognition
D. Beliefs about themselves and about mathematics.

A. The Knowledge Base

What you know is important.

(Doh!)

But, knowledge in itself is not enough. It's what you do with it that counts.

Think of a whole set of tools in a tool shop. What I might do with them, and what a craftsperson does, are very different!

A lot of knowledge is *inert.* Students can solve the problems we show them how to solve, and no more. They need to be *flexible* and *resourceful.*

B. Problem Solving Strategies

Here are some of the problem solving strategies described in George Pólya's book *How to Solve It:*

- draw a diagram

- look at cases

- solve an easier related problem…

The challenge:
These strategies may sound simple, but they’re not as easy to use as they sound.

For example, consider the strategy,

“If you can’t solve the given problem, try to solve an easier related problem and then exploit either the method or the result that you used.”

Steps in using a simple strategy like "Exploit an easier related problem"

1. Think to use the "strategy".
2. Know which version of the strategy to use.
3. Generate appropriate and potentially useful easier related problems.
4. Select the right easier related problem.
5. Solve it.
6. Be able to exploit it....

The Moral: The strategies are tough, and you need detailed training and lots of practice

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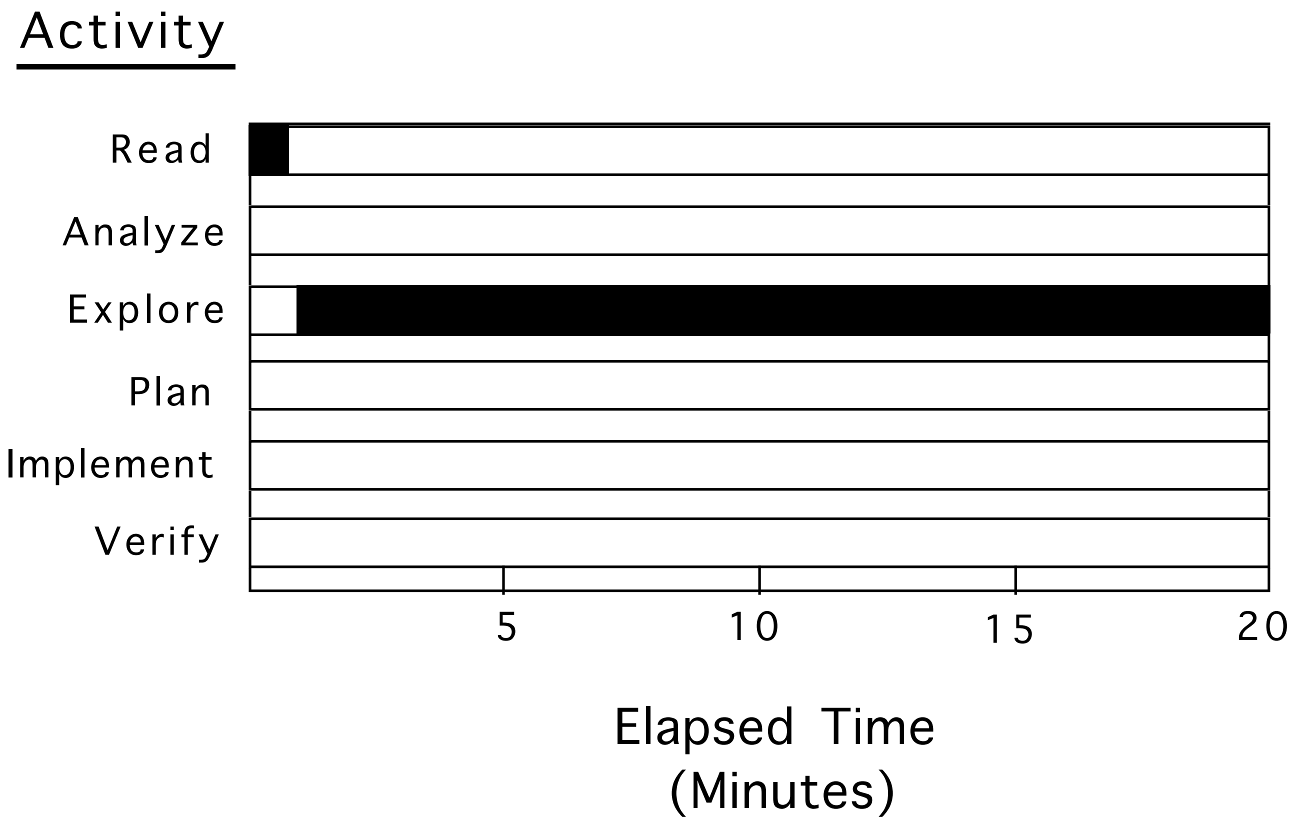
The Results

Students solved problems I couldn't.

C. “Control”: Monitoring and Self-Regulation, or Metacognition

What matters isn’t simply what you know – it’s how and when you use what you know!

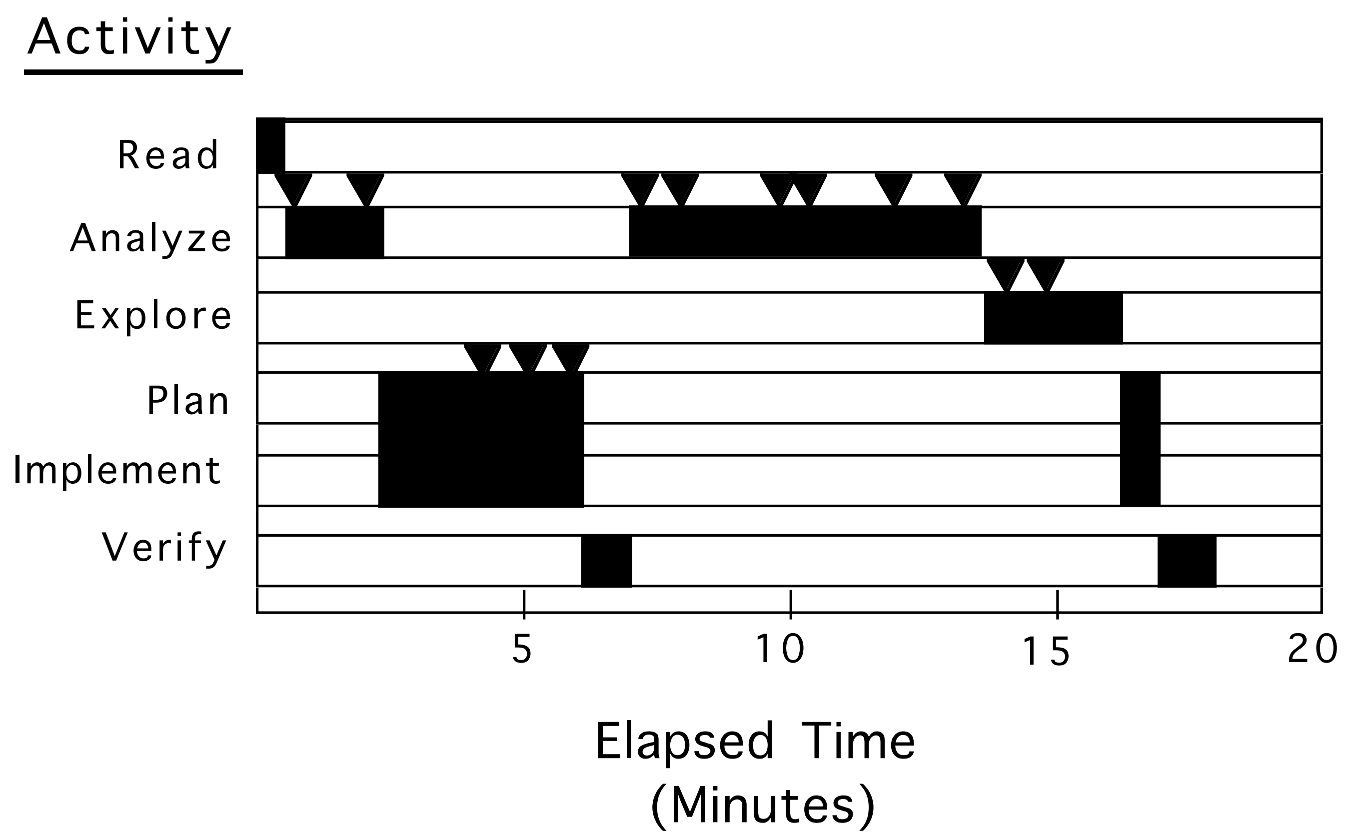
Here’s a *typical* graph of two students working a problem *that they knew enough to solve.*



Time-line graph of a typical student attempt to solve a non-standard problem.

A contrasting example:

A mathematician working a complex 2-part problem, and making very effective use of what he knows.



Time-line graph of a mathematician
working a difficult problem

I have these questions posted, and emphasize them:

What (exactly) are you doing?

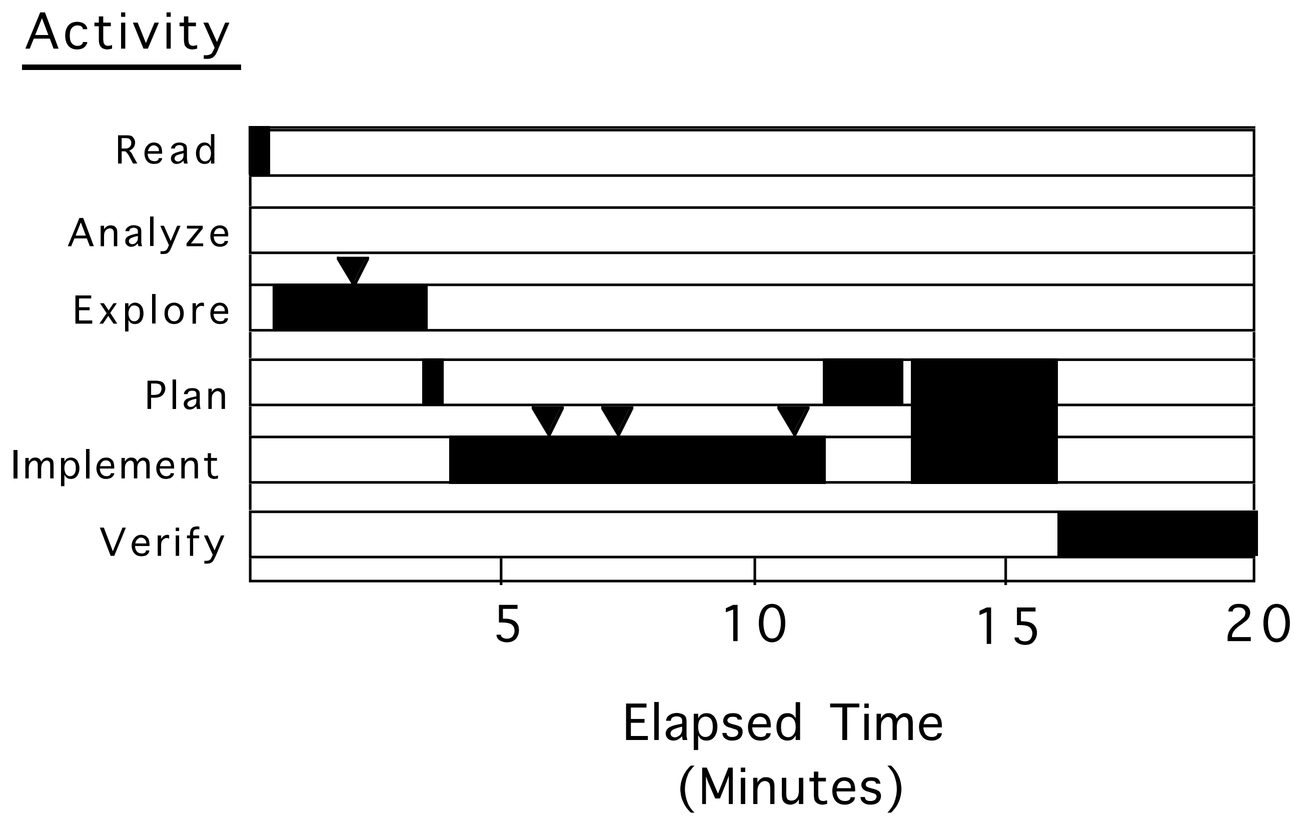
(Can you describe it precisely?)

Why are you doing it?

(How does it fit into the solution?)

How does it help you?

(What will you do with the outcome when you obtain it?)



Time-line graph of two students working a problem after the problem solving course.

D: Beliefs, and where they come from

U. S. National Assessment of Educational Progress

Carpenter, Lindquist, Matthews, & Silver, 1983

An army bus holds 36 soldiers. If 1128 soldiers are being bussed to their training site, how many buses are needed?

29% 31R12
18% 31
23% 32
30% other

Some Typical Student Beliefs about Mathematics

1. There is one right way to solve any mathematics problem.
2. Mathematics is passed on from above for memorization.
3. Mathematics is a solitary activity.
4. All problems can be solved in 5 minutes or less.
5. Formal proof has nothing to do with discovery or invention.
6. School mathematics has little or nothing to do with the real world.

Students develop their sense of mathematics (or any other subject matter) from their experience with it.

It is possible to create a culture of mathematical sense-making in the classroom, where students experience mathematics as a form of sense-making.
In such a context, they can develop the kinds of knowledge and beliefs that will enable them to be effective problem solvers.
That is the kind of environment one would hope to see in our mathematics classrooms. We owe it to our students.

Discussion

Part 2

What are the attributes of good problems, and instruction that makes the most of them?

A key point about problems…

Math is not simply about "answer getting."

The point isn't, "how do I use this technique to get an answer to the kinds of problems I've been shown how to solve?"

It's how do I make sense of this situation, using the mathematical tools at my disposal?

Good problems provide fertile grounds for developing this kind of understanding.

So, really good problems should:

- Be accessible (not require a lot of machinery)
- Be solvable a number of ways
- Illustrate important ideas
- <u>Not</u> have trick or mechanical solutions
- Support rich mathematical explorations and conversations.

Consider, for example, these two questions:

Problem 1:

Compute the mean, median, and standard deviation of these two distributions:

a. -3.5, .75, 1.5, 4.5, -.75, -2.5, 4.75, 2.75, .5, -1.5, 2.25, 9.25, 3.5, 1.25, -.5, 2.5, .5, 7.25, 5.5, 3;
b. 3.75, 4.5, 3, 5, 2.25, 1.25, .75, 3, -.5, 1.5, 3.5, 6, 4.5, 5.5, 2.5, 4.25, 2.75, 3.75, 4.75

Problem 2:

You work for a business that has been using two taxicab companies, Company A and Company B.

Your boss gives you a list of (early and late) "Arrival times" for taxicabs from both companies over the past month.

Your job is to analyze those data using charts, diagrams, graphs, or whatever seems best. You are to:

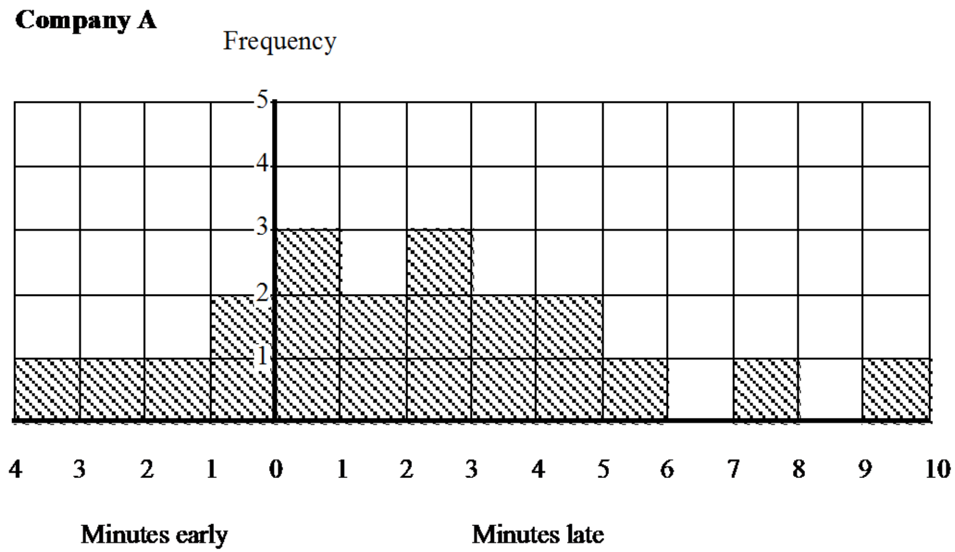
i. make the best argument that you can in favor of Company A;

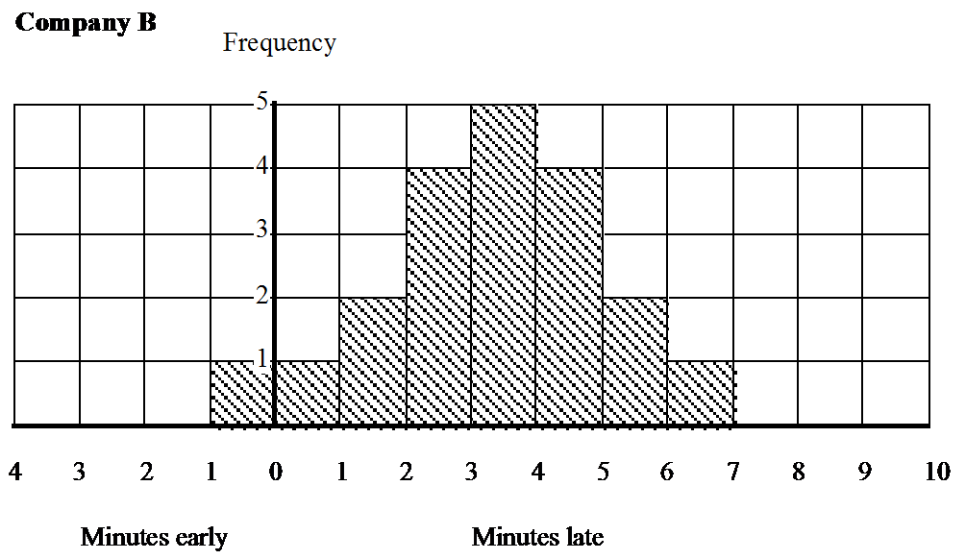
ii. make the best argument that you can in favor of Company B;

iii. write a memorandum to your boss that makes a reasoned case for choosing one company or the other, using the relevant mathematical tools at your disposal.

Company A		Company B	
3 mins 30 secs	Early	3 mins 45 secs	Late
45 secs	Late	4 mins 30 secs	Late
1 min 30 secs	Late	3 mins	Late
4 mins 30 secs	Late	5 mins	Late
45 secs	Early	2 mins 15 secs	Late
2 mins 30 secs	Early	2 mins 30 secs	Late
4 mins 45 secs	Late	1 min 15 secs	Late
2 mins 45 secs	Late	45 secs	Late
30 secs	Late	3 mins	Late
1 minute 30 secs	Early	30 secs	Early
2 mins 15 secs	Late	1 min 30 secs	Late
9 mins 15 secs	Late	3 mins 30 secs	Late
3 mins 30 secs	Late	6 mins	Late
1 min 15 secs	Late	4 mins 30 secs	Late
30 secs	Early	5 mins 30 secs	Late
2 mins 30 secs	Late	2 mins 30 secs	Late
30 secs	Late	4 mins 15 secs	Late
7 mins 15 secs	Late	2 mins 45 secs	Late
5 mins 30 secs	Late	3 mins 45 secs	Late
3 mins	Late	4 mins 45 secs	Late

The data may be analyzed and graphed as follows.





	Company A	Company B
Mean	2 mins 3 secs	3 mins 14 secs
Median	1 min 53 secs	3 mins 15 secs
Range	12 mins 45 secs	6 mins 30 secs
SD	3 min 11 secs	1 min 40 secs

Company A's cabs are earlier on average than Company B's, but they are less consistent in their arrival times.

It's better to order a cab from Company B - but order it for 5 minutes early, so it arrives when you need it.

For those of you who teach elementary school, consider these two tasks:

1. What happens when you add two even numbers? An odd number and an even number? Two odd numbers? Will it always happen? Can you show why?
2. Can you find a fraction between ½ and ¾? What about these two? (pick any 2 fractions.) Can you always do it?

But it’s not just the task, it’s what you do with it.

Consider this task:

Freight train A leaves the station traveling at 50 km per hour. Three hours later freight train B leaves the station on a parallel track traveling at 60 km per hour. How long does it take train B to catch up with train A?

Simple, right?

Just solve

$$50t = 60(t-3)\ldots$$

But remember that the catch-up time is (t-3).

Or, solve

$$50(t+3) = 60t,$$

and t is the time train B took.

A teacher recently told me this is a 2-minute problem. (OK, maybe 5, she said.)

I’ve worked on it with math majors for an hour.

I ask them to find a solution and make posters.

About half make tables!

Half of the rest draw graphs

The rest use algebra.

Here’s how the conversation goes…

The students who made tables got variants of this:

Hours	Train A	Train B
1	50	
2	100	
3	150	
4	200	60
5	250	120
6	300	180
7	350	240
8	400	300
9	450	360
10	500	420
11	550	480
12	600	540
13	650	600
14	700	660
15	750	720
16	800	780
17	850	840
18	**900**	**900**
19	950	960
20	1000	1020

Hours	Train A	Train B
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4	200	120
5	250	180
6	300	240
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10	500	480
11	550	540
12	600	600
13	**650**	**660**
14	**700**	**720**
15	750	780
16	800	840
17	850	900
18	900	960
19	950	1020
20	1000	1080
	1050	

Hours	Train A	Train B
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	100	
	150	
1	200	60
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4	350	240
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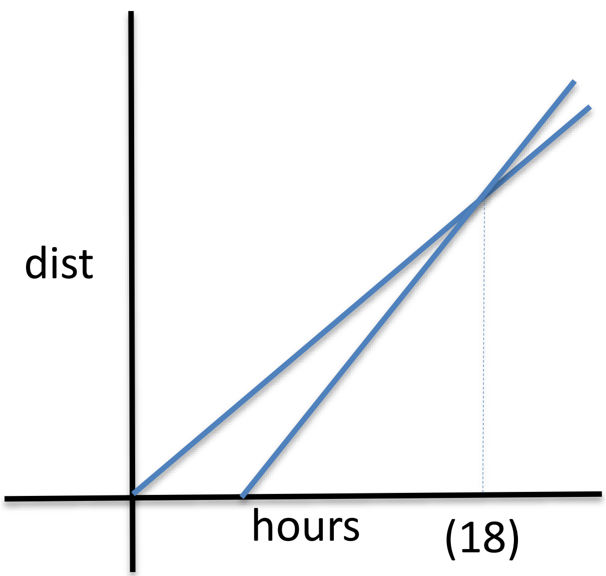
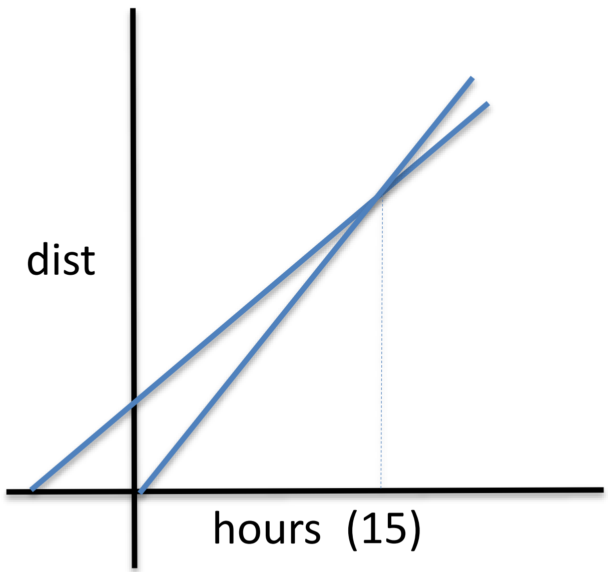
7	500	420
8	550	480
9	600	540
10	650	600
11	700	660
12	750	720
13	800	780
14	850	840
15	**900**	**900**
16	950	960
17	1000	1020
18	1050	1080
19		
20		

Those who made graphs got variants of this… or this:



Or



Those who did algebra got variants of this... or this:

$50t = 60(t-3)$... Or $50(t+3)t = 60(t)$

And then the fun began.

- Which table is right (or preferable), and why?
- Where do you see train B catching up?
- Can you predict where? Why?
- Which graph is right (or preferable), and why?
- Where can you see everything we saw in the tables, in the graphs? (including "catching up")
- Which equation is right (or preferable), and why?
- Where can you see everything we saw in the tables and graphs, in the equations??

So again, the issue is…

NOT, “what’s the answer,” but

“What opportunities for sense-making, including making connections, does a problem offer us,” and

“How can we make rich use of a problem to see and understand the underlying mathematics?”

Discussion

Part 3

What are the attributes of “powerful instruction” – instruction that produces students who are powerful thinkers and problem solvers?

We’ll begin with this question.

If you had 5 things to focus on in order to build classrooms that produce students who are powerful thinkers, what would they be?

Why 5 (or fewer)?

It’s as many as most folks can keep in mind. (In fact, it may be too many to work on at one time.)

If you have 20, you might as well have none. People can’t keep that many things in their heads, and long check lists don’t help. What matters is what people can act on, in teaching and coaching.

What properties should those 5 things have?

They're all you need (there's nothing essential missing).

They each have a certain "integrity" and can be worked on in meaningful ways.

Their framing supports professional growth.

You’re about to meet the
Teaching for Robust Understanding
of Mathematics
(TRU Math)
framework

The Five Dimensions of Mathematically Powerful Classrooms

The Mathematics	Cognitive Demand	Access to Mathematical Content	Agency, Ownership, and Identity	Formative Assessment
The extent to which the mathematics discussed is focused and coherent, and to which connections between procedures, concepts and contexts (where appropriate) are addressed and explained. Students should have opportunities to learn important mathematical content and practices, and to develop productive mathematical habits of mind.	*The extent to which classroom interactions create and maintain an environment of productive intellectual challenge conducive to students' mathematical development. There is a happy medium between spoon-feeding mathematics in bite-sized pieces and having the challenges so large that students are lost at sea.*	*The extent to which classroom activity structures invite and support the active engagement of all of the students in the classroom with the core mathematics being addressed by the class. No matter how rich the mathematics being discussed, a classroom in which a small number of students get most of the "air time" is not equitable.*	*The extent to which students have opportunities to conjecture, explain, make mathematical arguments, and build on one another's ideas, in ways that contribute to their development of agency (the willingness to engage mathematically) and ownership of the content, resulting in positive identities as doers of mathematics.*	*The extent to which the teacher solicits student thinking and subsequent instruction responds to those ideas, by building on productive beginnings or addressing emerging misunderstanding s . Powerful instruction "meets students where they are" and gives them opportunities to move forward.*

Before proceeding, it's ESSENTIAL to understand:

TRU is NOT a tool or set of tools.

TRU is a perspective regarding what counts in instruction, and

TRU provides a language for talking about instruction in powerful ways.

With this understanding, you can make use of any productive tools wisely.

But, we have tools.

(of course.)

TRU contains and aligns with a large set of tools produced by the Mathematics Assessment, Algebra Teaching Study, and TRU-Lesson Study Projects.

Tools to Support Powerful Classroom Instruction

The Mathematics Assessment Project has produced 100 "Formative Assessment Lessons" (FALs) to help teachers engage in "diagnostic teaching."

By the time I give this talk, there will be more than 6,000,000 lesson downloads.

Tools to Capture Student Understandings

Since 1991, the

Mathematics Assessment Project

has been producing standards-based assessments.

These assessments (the Balanced Assessment or MARS tests) have been used in a variety of studies as a robust measure of mathematical thinking and problem solving.

Tools for the Collaborative Improvement of Teaching

The ***TRU Conversation Guide*** and the ***TRU Observation Guide are*** designed to help teachers, coaches, and Professional Learning Communities work on each of the 5 dimensions in depth.

A Tool for Planning for and Reflecting on Teaching

The ***TRU Math Conversation Guide*** is designed to foster reflective conversations about instruction.

Frame each dimension with questions:

The Mathematics

How do mathematical ideas from this unit/course develop in this lesson/lesson sequence?

Cognitive Demand

What opportunities do students have to make their own sense of mathematical ideas?

Access to Mathematical Content

Who does and does not participate in the mathematical work of the class, and how?

Agency, Ownership, and Identity

What opportunities do students have to explain their own and respond to each other's mathematical ideas?

Formative Assessment

What do we know about each student's current mathematical thinking, and how can we build on it?

. . . and expand them.

Before a lesson, you can ask:

- How can I use the five dimensions to enhance my lesson planning?

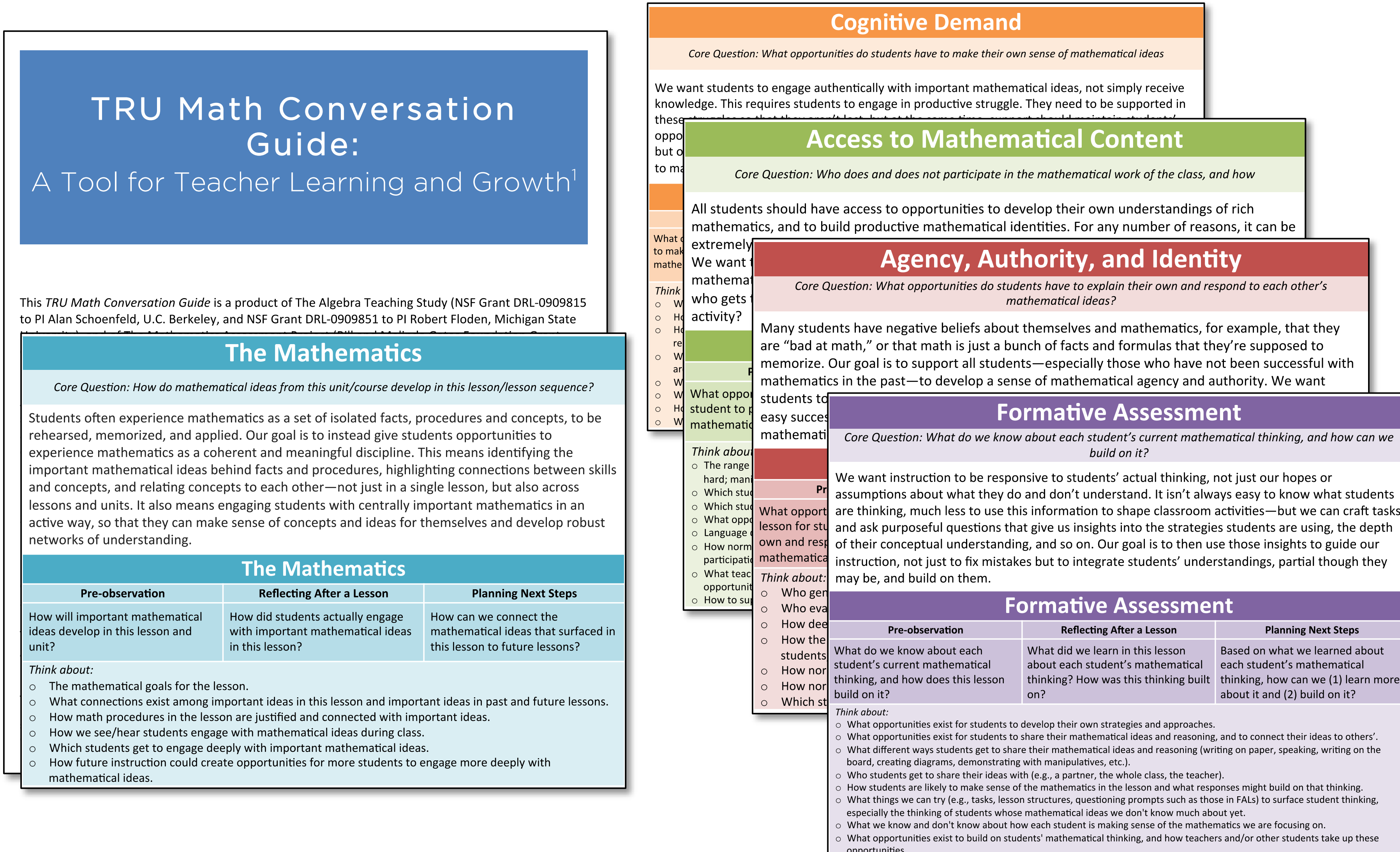
After a lesson, you can ask:

- How well did things go? What can I do better next time?

Planning next Steps, you can ask:

- How can I build on what I've learned?

The TRU Conversation Guide



…and, to support collegial observations

(not just in math, but every discipline)

we offer the TRU Math observation guide:

The TRU Observation Guide

The TRU Observation Guide:

A Tool for Teachers, Coaches, and Professional Learning Communities

THE CONTENT

The extent to which the central disciplinary ideas and methods, as represented by State or National Standards, are present and embodied in instruction. Students should have opportunities to learn important content, and to develop productive disciplinary habits of mind. Individual lessons should be part of a coherent whole, providing opportunities for students to grapple meaningfully with those ideas.

Students…	Teachers…
• Engage with grade level content in ways that highlight important information, concepts, and methods • Have opportunities to develop productive disciplinary habits of mind • Have opportunities to reason about disciplinary issues, both orally and in writing	• Highlight important ideas and provide opportunities for students to engage with them. • Support the purposeful use of academic language and other representations central to the discipline • Use materials or assignments that center on key ideas, connections, and applications • Support students in seeing the discipline as being coherent, connected, and comprehensible.

What are the big ideas in this lesson? How do they connect to what has come before, and/or establish a base for future work? How do students engage with them? How might their engagement support deeper engagement with the concepts, and stronger development of disciplinary habits of mind?

Goal: Students work on core disciplinary issues in ways that enable them to develop conceptual understandings, tie procedures to underlying concepts in clear ways, apply the mathematics they have learned in relevant contexts, and work to develop problem solving and reasoning skills.

COGNITIVE DEMAND

The extent to which classroom interactions create and maintain an environment of productive intellectual challenge conducive to students' deepening understanding of disciplinary content and practices. There is a happy medium between spoon-feeding content in bite-sized pieces and having

ACCESS AND EQUITY

The extent to which classroom activity structures invite and support the active engagement with core content by all of the students in the classroom. No matter how rich the content being discussed, a

AGENCY, OWNERSHIP, AND IDENTITY

The extent to which students have opportunities to explore, conjecture, explain, make arguments and build on one another's ideas, in ways that contribute to their development of agency (the capacity and willingness to engage academically) and to their ownership over the content, resulting in positive

FORMATIVE ASSESSMENT

The extent to which the teacher solicits students' mathematical thinking and subsequent instruction responds to those ideas, by building on productive beginnings or by addressing emerging misunderstandings. High quality instruction "meets students where they are" and gives them opportunities to move forward, both as guided by the teacher and in student-to-student interactions.

Students…	Teachers…
• See errors as a chance for new learning • Explain their thinking, even if somewhat preliminary • Work to justify their findings, and to apply their ideas • Consistently conduct assessments of their own work and the work of peers • Provide specific and accurate feedback to fellow students • Make use of feedback in revising their own work	• Use tasks and lessons that provide opportunities for student engagement, and for eliciting diagnostic and formative information about student understandings • Create safe climates in which students feel free to express their ideas and understandings • Employ a variety of formative assessment techniques and use results to inform adjustments to content and process, providing opportunities for re-engagement and revision • Provide feedback to students that is timely and prompts students to make active use of that information in their learning • Support student interactions over content that address emerging understandings • Create opportunities for students to reflect, individually and collaboratively, on their understandings and on their learning.

What opportunities exist to build on students' thinking? How do teachers and/or other students take up these opportunities?

Goal: Each student's learning experience is maximized through the strategic and flexible use of a wide range of assessments throughout instruction, and subsequent interactions centering on them.

Resources:

The TRU Math Suite, this talk, and supporting documents are available on

The Algebra Teaching Study web site:

http://ats.berkeley.edu/

under “tools” and “publications” tabs

and

The Mathematics Assessment Project web site:

http://map.mathshell.org/

under “TRU Math Suite” tab

(Just Google the project names.)

Discussion