Chapter 12 Problem Solving in a 21st Century Mathematics Curriculum

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Problem solving makes a wonderful banner under which to march as long as no one looks too closely at what others mean by the term.

(Kilpatrick, 1981, p. 2)

Introduction

Research on problem solving in the mathematics curriculum has spanned many decades, yielding pendulum-like swings in recommendations on various issues. Ongoing debates concern the effectiveness of teaching general strategies and heuristics, the role of mathematical content (as the means versus the learning goal of problem solving), the role of context, and the proper emphasis on the social and affective dimensions of problem solving (e.g., Lesh & Zawojewski, 2007; Lester, 2013; Lester & Kehle, 2003; Schoenfeld, 1985, 2008; Silver, 1985). Various scholarly perspectives—including cognitive and behavioral science, neuroscience, the discipline of mathematics, educational philosophy, and sociocultural stances—have informed these debates, often generating divergent resolutions. Perhaps due to this uncertainty, educators' efforts over the years to improve students' mathematical problem-solving skills have had disappointing results. Qualitative and quantitative studies consistently reveal mathematics students' struggles to solve problems more significant than routine exercises (OECD, 2014; Boaler, 2009).

Another perspective on problem solving considers the demands of modern life and work. We acknowledge that preparation for adult work and life is not the only goal of mathematics education. We contend, however, that for the vast majority of students (who will not become academic mathematicians), enhancing their opportunities and performance in work and life should indeed be the main purpose for mathematics education. Worldwide, linking mathematics education and workplace preparedness has become a central policy theme (Grubb & Lazerson, 2004; Mehta, 2013; Miller, n.d.), while linking with life enhancement is less so. Nevertheless, insufficient effort has been made to move beyond this policy rhetoric—to critically examine the mathematical demands of 21st-century work and life, and to consider how these demands should reshape mathematics education. This chapter aims to contribute to this effort. We examine how employers, workers, economists, and other scholars portray the problem-solving demands of modern work and life, and the contributions of schooling. We then consider how certain historical problem-solving debates could be resolved if the overriding purpose of mathematics instruction were to prepare students to meet the demands of work and life today.

One difficulty immediately arises when investigating problem solving in mathematics education: Numerous interpretations of *problems* and *problem solving* have been offered over the years with no universally accepted definitions (English & Sriraman, 2010; Lesh & Zawojewski, 2007; Lester, 2013; Schoenfeld, 2013; Toerner, Schoenfeld, & Reiss, 2008; Zawojewski, 2010). The domain of problem solving is broad, resulting in the myriad approaches to defining it. In 1981, Kilpatrick complained that the "imprecise and indiscriminate use" of the terms *problem* and *problem solving* "allows numerous sins to be committed in their name" (p. 2), and the situation seems hardly improved today.

Traditionally, problems have been defined as tasks in which the solver does not know how to arrive at an answer. Lester (2013) reviews numerous examples of this sort of definition, such as Duncker's: "A problem arises when a living creature has a goal but does not know how this goal is to be reached" (1945, p. 1). Newell and Simon (1972) echoed this notion of blockage but included a motivational aspect: "A person is confronted with a problem when he wants something and does not know immediately what series of actions he can perform to get it" (p. 72). Some more recent definitions of mathematical problem solving still refer to the uncertainty of solution, such as Mamona-Downs and Downs's (2013) notion of "engagement on any mathematical task that is not judged procedural or the student does not have an initial overall idea how to proceed in solving the task" (p. 139). Other scholars recognize the breadth of the notion, as can be seen in English and Sriraman's (2010) simple statement regarding their use of the term *problem solving*: "in a broad sense to cover a range of activities that challenge and extend one's thinking" (p. 263). Also offering a broad, albeit more comprehensive, view is Hegedus's definition:

> We take a very broad view of what is mathematical problem-solving viewing it as an enterprise of collaborative investigation where multiple approaches are valid. It is not just about solving a specific problem, which has a specific answer or application into the real world, but rather it is an investigation that might have multiple approaches and where students can make multiple observations.
>
> (2013, p. 89)

These newer interpretations of problems and problem solving reflect dissatisfaction with the traditional notions and their unhelpfulness for the teaching of problem solving (Lester, 2013).

Taking the perspective of preparation for adult work and life, another ambiguity emerges: Should *problems* in mathematics education refer to *mathematical* problems (posed, solved, and concluded in the domain of mathematics) or to real-world problems and problems in other domains (such as science or business) that can be solved by applying mathematics? In our research for this chapter, we tried to stay open with regard to the meaning of *problem solving in mathematics education*, to avoid arbitrarily constraining our interpretations of what 21st-century demands might require of mathematics education.

We structure the remainder of our chapter as follows. First, we review briefly some key debates in mathematical problem-solving research of past decades. We then review the literature about the demands of modern work and life. Here, we examine drivers of change in the workplace and everyday life, the nature of quantitative problems that need to be solved in these changing contexts, and the competencies required in doing so. Finally, we return to the key debates to discuss possible resolutions suggested by the research about problem solving in the 21st century.

Debates (and Disappointments) of Mathematical Problem-Solving Research

Promoting mathematical problem solving has been a long-standing, often contentious endeavor. The importance of problem solving in the mathematics curriculum has been universally recognized, and scholars have advanced numerous strategies for building students' competencies in solving multiple problem types. The 1980s was to be the "decade of problem solving," with the National Council of Teachers of Mathematics' (NCTM) *An Agenda for Action* recommending the mathematics curriculum be organized around problem solving (1980, p. 2). Recent years have seen a resurgence of interest in problem solving among mathematics-education researchers, as evidenced by the many publications devoted to numerous issues pertaining to problem-solving theory and practice, both in school and beyond. For example, the 2013 special issue of *The Mathematics Enthusiast* (Vol. 10, nos. 1–2) aimed to address "the current trends in problem solving research and … the main results that influence teachers' practices and curricula design" (Moreno Armella & Santos-Trigo, 2013, p. 4). In another journal special issue (*ZDM*, Vol. 39, nos. 5–6), Toerner, Schoenfeld, and Reiss (2008) edited a review of problem-solving developments around the world. And a recent special issue of *Educational Studies in Mathematics* (Vol. 83, no. 2013) and an edited book, *Problem Posing: From Research to Effective Practice* (Singer, Ellerton, & Cai, 2013) were devoted entirely to the topic of problem *posing*.

Unfortunately, this decades-long focus on mathematical problem solving, while yielding important insights into the phenomenon, has not produced clear guidance for educational practice. Questions about how to promote problem-solving competency remain; indeed, we contend that they have become more perplexing in light of proliferating interpretations of problems and problem solving, the recent emphasis in many countries on equitable education for a greater range of students, and the changing demands of work and society. In particular, certain unresolved debates appear to impede our forward movement. We review some of those debates here.

Teaching Problem Solving versus Teaching Mathematics *through* Problem Solving

Should the overarching goal of using problems in the mathematics classroom be to teach problem solving per se, or to teach mathematical content, using problem solving as a vehicle? Some scholars (e.g., Anderson, 2014) blame disappointing student gains on the traditional treatment of classroom problem solving, where it is independent of, and isolated from, the development of core mathematical ideas, understandings, and processes. In school, problem solving often takes the form of application ("story") problems at the end of the textbook chapter, positioning it as an add-on task, presumably to promote the ability to apply already-learned content. Such problems rarely serve either the purpose of teaching problem solving or building or deepening the knowledge of that content (Anderson, 2014). But the limited research attention to how concept development might be accomplished *through* problem solving indicates that problem solving has not been seen as playing a central role in the curriculum but rather has been pushed to the periphery (Rigelman, 2013). Also needed are studies that explore whether both goals can be accomplished at once, examining the impact of problem-driven conceptual development on the development of problem-solving competencies (Lester & Charles, 2003; Schoen & Charles, 2003). In sum, while more recent scholarship favors problem solving as a means for developing mathematics-content understanding as opposed to an end in itself, the debate is far from settled.

The Effectiveness of Heuristics and General Skills

Closely related to the debate about the purpose of problem solving in the classroom is a second debate about how to teach students to solve problems. Earlier scholarship rested on the notion of problem solving as a general ability (or ability set) that could be developed across content areas or even in a decontextualized manner. Perhaps the most contentious facet of this debate has been the efficacy of teaching general strategies and heuristics—the tools of an "expert" problem solver—instigated largely by Polya's seminal book, *How to Solve it* (1945). This book has long been regarded as a valuable resource for improving students' abilities to solve unfamiliar problems by offering a list of steps and solution strategies to take when "stuck." Despite some evidence that such tools can contribute to successful problem solving, they nevertheless appear not to have delivered the improvements in problem solving that educators envisioned many years ago (e.g., Lesh & Zawojewski, 2007; Lester & Kehle, 2003; Schoenfeld, 2013; Silver, 1985).

Other general competencies have also been associated with problem solving. Metacognition (the reflection of the solver on his or her thinking and solving processes) is presumed to influence problem solving, with more sophisticated levels of self-awareness and explicitness about strategies being associated with greater success in solving problems (Kapa, 2001; Schneider & Artelt, 2010). Over the years, numerous instructional interventions have been developed and implemented to enhance metacognition as an indirect means of improving problem-solving competence (e.g., Goos, Galbraith, & Renshaw, 2002; Kramarski, Weisse, & Kololshi-Minsker, 2010). Social skills such as collaboration and communication have also been linked to problem-solving competence and, again, targeted directly with instructional interventions (Goos & Galbraith, 1996; McKenna & Agogino, 2004).

Overall, there appears little evidence to suggest that improving these general skills or heuristics leads to greater success in solving problems (mathematical or otherwise) (Lester, 2013), though other positive outcomes surely result. One explanation for this limited success is that these general skills are often presented as a collection of separate entities to be learned and applied, without students fully knowing and understanding why, when, and how to do so (e.g., English & Sriraman, 2010; Lesh & Zawojewski, 2007; Lester, 2013). Another explanation is that problem-solving skills and heuristics, initially conceived to be used interactively with students engaged in authentic problem solving, are often incorporated into textbooks didactically and thus reduced to procedural algorithms (Stanic & Kilpatrick, 1989).

The Role of Context and Authenticity

Mathematics educators frequently debate the role of realistic contexts in teaching. Students' difficulties in applying mathematical concepts and abilities (that they presumably have learned in school) outside of school, or in other classes, such as those in the sciences, have been amply documented (de Abreu, 2002; Greiffenhagen & Sharrock, 2008; Nunes, Schliemann, & Carraher, 1993). A prevailing explanation for these difficulties is the context-specific nature of learning and problem solving. That is, problem-solving competencies that are learned in one situation take on features of that situation; transferring them to a new problem in a new context poses challenges (Lobato, 2003; Hohensee, 2014). This view of problem solving would also explain why acquiring general (heuristic, metacognitive, and social) skills might do little to improve problem-solving competence; indeed, it challenges the existence of "problem-solving competence" as a unitary phenomenon. One resolution is to situate mathematics learning in real-world problem-solving contexts, although the problem remains that mathematics learned in one context does not easily transfer to other contexts. Additional concerns have been raised regarding the equitability of contextualizing mathematics instruction. Lubienski (2000) found children from low-income households less able to access the mathematics in a contextualized curriculum, while Cooper and Dunne (1998) showed low-income students scoring more poorly on contextualized assessment questions. Both studies concluded that the context presented a distraction that higher income students knew to ignore. Finally, importing real-world problems and contexts into the classroom necessarily reduces their authenticity, for pedagogical and logistical reasons. As Bakker, Kent, Derry, Noss, and Hoyles (2008) warned, we cannot simply reproduce workplace experiences within the classroom in the hope of increasing students' chances of success beyond school. More research is needed to settle questions about whether teaching mathematics through real-world problem contexts improves students' abilities to solve a range of problems in adulthood and, if so, how authentic those contexts must be.

What Mathematical Content to Teach

A more general debate that overlaps with the issue of problem solving concerns what mathematical content is most important to teach. This question, too, can be answered from many perspectives, including the perpetuation of the discipline, the preparation of future mathematicians, readiness for future mathematics or other classes, and personal intellectual development or enjoyment. Again, the perspective we take here is preparation to meet the demands of modern work and life, with full recognition that this is only one of many valid goals.

In the next section, we examine how employers, workers, economists, and other scholars portray the problem-solving demands of modern work and life and the contributions of schooling, as a means of shedding new light on these classic debates.

The Demands of 21st-Century Work and Life

Drivers of Change

The very phrase "21st-century demands" implies a view that life and work today significantly differ from life and work even a few decades ago, in ways that alter cognitive requirements and obligate new educational priorities. Before examining those new requirements and priorities, we first ask what has driven change in life and work and what is the basis for claims that their requirements are different.

Those who believe 21st-century life and work has changed significantly point to several sources. Technological advances and ubiquity are perhaps the most commonly cited drivers of change (Brynjolfsson & McAfee, 2011; Goldin & Katz, 2008; Handel, in press). In the workplace, computers and robots now accomplish routine or manual tasks that once required human actors (Autor, Levy, & Murnane, 2003; Partnership for 21st-Century Skills [P21], 2008). This development, in turn, purportedly requires workers to have higher-level problem-solving skills (Kaput, Noss, & Hoyles, 2008; Hoyles, Noss, Kent, & Bakker, 2010; P21, 2008). Multiple explanations have been offered for the mechanism by which technology elevates cognitive demands on the workforce as a whole. Automation by computers or robots may be *replacing* low-cognitive-level jobs, leaving only higher-level jobs remaining. Alternately, technology may be *transforming* what were once low-level jobs, because working *with* technology, or coping with technological change requires higher levels of cognition (Schultz, 1975; Welch, 1970 [both cited in Pellegrino & Hilton, 2012]). Or the mechanism may be less direct: some research finds that company-level technology investments yield productivity gains only when accompanied by organizational changes such as new strategies, processes, practices, and structures (P21, 2008). Thus, technology may be altering the cognitive demands on workers by changing their organizational roles, engaging them, for example, in self-managed teams, information sharing, and/or decentralized decision making.

Which, if any, of these mechanisms accurately ties technological advances to increased cognitive demands at work is uncertain. As Brynjolfsson and McAfee write, “Digital technologies are one of the most important driving forces in the economy today. They’re transforming the world of work and are key drivers of productivity and growth. Yet their impact on employment is not well understood, and definitely not fully appreciated” (2011, p. 9). Handel’s (in press) study of U.S. workers challenges both the job-replacement and job-transformation explanations. Furthermore, changes in organizational structures and practices that elevate cognitive demands on workers, such as flatter hierarchies and greater use of teams (Tucker, 2013), may have other causes than technology. Globalization and intensifying international competition are offered as other major drivers of elevated cognitive demands in the workplace (Hoyles et al., 2010). Most employers surveyed by the Partnership for 21st-Century Skills felt global competitiveness had shifted the importance of certain skills and competencies in their companies. Other change drivers—some of which are interdependent with technology, globalization, and each other—include the transformation from a manufacturing to a service or information economy (P21, 2008; Reich, 1991[cited in Pellegrino & Hilton, 2012]), the rapid pace of change in business (P21, 2008), mass customization of products, and elevated standards for communication with customers (Hoyles et al. 2010).

Despite these economic changes, there is some debate about whether the cognitive demands of the workplace, overall, are actually rising. Levy and Murnane (2004 [cited in Pellegrino & Hilton, 2012]) argue that the modern workplace increasingly requires the ability to solve nonroutine problems, as well as complex communication competencies and verbal and quantitative literacy. Yet a meta-analysis by Bowles, Gintis, and Osborne (2001), as well as other studies reviewed by Pellegrino and Hilton (2012), show small to no correlations between scores on basic cognitive tests and earnings since the 1970s, suggesting that the labor-market demand for cognitive competencies has been static. Also debated is whether changes in workforce needs, if they do exist, constitute a crisis, as many proclaim. In the U.S., for example, STEM fields may be enjoying high job growth (Langdon, McKittrick, Beede, Khan, & Doms, 2011) but experts (e.g., Atkinson & Mayo, 2010; Salzman, Kuehn, & Lowell, 2013) differ in their assessments of employers’ ability to fill those new positions. Other experts (Barton, 2000; Pellegrino & Hilton, 2012) also note an elevated premium on college degrees but question the reason. Rather than requiring college-level skills and knowledge, employers may seek a college degree only as a means of screening for basic skills, persistence, or work ethic (Murnane & Levy, 1996; Vedder, Denhart, & Robe, 2013).

Perhaps the new environmental condition that matters most to education is ideational. Mehta (2013) describes how the 1983 report *A Nation at Risk* linked—to a degree not seen before—schooling to individual and national economic success, thus engendering a new paradigm. This paradigm presumed that schools (not social forces) should be held responsible for academic achievement, were substantially underperforming in this role, and could be compelled to improve through monitoring by standardized cognitive tests—presumptions that have opened the door for and legitimized a dramatic shift from local to federal control of U.S. public schools. The implication of this new paradigm for our discussion is that notions of elevated cognitive demands on today's workers may not reflect actual economic or technological developments as much as a new perspective on the purpose of schooling (and policies that embody this perspective) that directly ties education to the quality of the workforce.

Overall, despite general agreement that broad economic and technological change has occurred in the past few decades, its impact on employment and its cognitive demands are not well established. Handel (in press) sums up the situation: "Researchers have only a cloudy sense of the levels and kinds of job skill requirements, rates of change, the dimensions along which job skills are changing, and the interrelationships between skills, technology, and employment involvement" (p. 3).

Unsurprisingly, given a current policy environment that prioritizes economic gain among the purposes of education, less research has targeted the changed demands of 21st-century life outside of work. A major source of such scholarship is Decision Research (www.decisionresearch.org), a nonprofit organization that investigates human judgment, decision making, and risk. Multiple studies from this group document recent movement in the areas of health care and personal finance towards greater consumer decision making, at the same time that available information about these areas is burgeoning. In health care, "Coverage choices are becoming more complicated and varied, health delivery systems more complex, and evidence of provider quality and treatment efficacy more transparent. Consumers therefore require more knowledge and greater skill to take full advantage of new sources of information and to make appropriate choices" (Hibbard, Peters, Dixon, & Tusler, 2007, p. 380). In one study, 50% of people seeking information about cancer first consulted the Internet; only 25% first consulted a doctor (Nelson, Fagerlin, & Peters, 2008). Much of this health-care and financial information is represented by statistics (e.g., regarding risks and benefits) and graphs (e.g., survival and mortality curves), as well as in complex documents and forms (e.g., from insurers). One's ability to understand this information has obvious consequences for one's health and well-being (Nelson et al., 2008).

In 1988, Davis observed that virtually all aspects of modern life had become mathematized, including driving, warfare, and even aesthetic judgment. He argued that citizens now needed not technical mathematical skills as much as an understanding of the ways mathematics shaped their lives, so that they could participate knowledgeably in social decisions rather than ceding control to a mathematically expert elite. In 2015, Davis's observation has only become truer. (Indeed, this sort of socio-mathematical savvy might allow people to more critically analyze policy rhetoric about the economic imperatives for increased schooling or cognitive skill!)

Problems Faced in 21st-Century Life and Work

Whether or not the demands of 21st-century work and life have changed considerably, and regardless of the reason, it is still meaningful to ask about the nature of the problems that need to be solved there. Most investigations into the cognitive requirements of the modern workplace are general: asking large groups of employers what they desire from workers; or trying to correlate levels of schooling with employment, earnings, or national productivity. Smaller-grained studies of the kinds of intellectual problems that arise in work (or everyday life) are few and, by necessity, highly context specific. Such studies are usually ethnographic and thus, while providing a thorough characterization of problems in particular workplaces or everyday settings, are difficult to generalize across adult activity.

Much of the ethnographic research about the kinds of problems modern workers need to solve has been conducted by the Techno-Mathematical Literacies in the Workplace Project. Between 2003 and 2007, Hoyles, Noss, Kent, Bakker, and colleagues followed midlevel workers in IT-intensive settings: five manufacturing companies and two finance companies. While the problems these workers solved on a regular basis varied considerably, commonalities were observed within and across companies. All of the work centered on highly mathematized processes, for example, statistical process control (SPC) in manufacturing and the calculation of interest rates in finance. Graphs, charts, spreadsheets, and computer simulations displayed the input variables and output data for these processes. Everyday problems involved the impact of changes in input variables on output data, requiring workers to interpret these technological displays. The research team coined the term *techno-mathematical literacies* (TmL) to capture the ways that mathematical processes were understood with and through technological representations. Hoyles et al. (2010) observed that in these workplaces, "Calculation and basic arithmetic are of subsidiary importance compared to a conceptual grasp of how, for example, process improvement works, how graphs and spreadsheets may highlight relationships, and how systematic data may be used with powerful, predictive tools to control and improve processes" (p. 168). Unfortunately, many workers lacked sufficient TmL to solve their everyday problems effectively. The team concluded that the workers' "major skills gap" could be closed not with more mathematical training (e.g., to understand the algebra in which the processes were formally described) but with a deeper understanding of the mathematical models underlying the processes. Bakker et al. (2008) further investigated the nature of problem solving engaged in by workers using SPC, in contrast to the statistical reasoning required in school tasks. They found that both forms of problem solving aim for generalization, use data as evidence, employ probabilistic language, and compare data against models. But SPC involves generalizing about a *process* (rather than a population, as in school tasks) and its goal is (a decision about) action, to reduce output discrepancies. Thus, SPC requires *abductive* inference (explaining data anomalies as results of process events or conditions) rather than *inductive* (simply predicting data patterns). Also, unlike school tasks, which often entail suppressing context, interpreting SPC data relies on context (e.g., cost, knowledge of process).

Gainsburg's (2006, 2007a, 2007b) findings from her ethnographic study of structural engineers echo the findings of the TmL team. The phenomena at the center of the engineers' problem solving were not processes per se but structures and their behavior. However, the need to understand underlying models that were represented mathematically (and technologically) was just as crucial. Indeed, in Gainsburg's assessment, "The heart of the intellectual work of structural engineers is the application of mathematical representations and procedures to solve design problems, which usually requires the selection, adaptation, or creation of a model" (2007b, p. 38). In structural engineering, mathematical models are unavoidable because the structures do not yet exist. A main source of problems is the complexity and uniqueness of each building, which preclude the simple application of established procedures. Structural engineers need a deep understanding of structural behavior, combined with conceptual fluency with usually basic mathematics, to create models that accurately represent the proposed building or elements. As with midlevel manufacturing and financial work, the problems that structural engineers must solve are, at root, about prediction, and the tools that support that prediction are mathematical models and processes.

It is important to note that not only do these ethnographic findings not necessarily generalize across workplaces, they do not even represent the work of all employees within these fields. For example, a study by Kent and Noss (2002) of a much larger structural-engineering firm than the ones studied by Gainsburg painted a different picture: Here, the creation of mathematical models was assigned to mathematical specialists and not generally handled by engineers (also Dudley, 2010). Similarly, in the manufacturing firms in the TmL project, higher-level employees interacted with the processes in ways that required more formal mathematical understanding and manipulation, while lower-level workers presumably never solved problems involving mathematically described processes.

Handel (in press) conducted a rare example of a broader investigation about the nature of problems encountered across a spectrum of work settings. In his survey of 2,000 workers across a range of U.S. workplaces, only 22% of the respondents reported having to solve "hard" problems "often" in their jobs; about 33% said they rarely or never had to do so. While the frequency of hard problems did not vary greatly by broad occupational type, it was somewhat correlated with the level of formal education a job required. The contrast between Handel's findings and those of the TmL team and Gainsburg is striking. One explanation might be that the latter focused on unusually challenging jobs. A different explanation might be methodological: people are known to be poor describers of their own activity. In particular, when reporting on their mathematical problem solving, people default to school-type characterizations of mathematics (formal and algorithmic), which are rarely evident in their work (Hoyles, Noss, & Pozzi, 2001). Ethnographers generally take a broader view of mathematics and "see" the same people using mathematics to solve everyday problems. It must also be noted that the TmL team detected problems that could have been (better) solved with more significant understanding and quantitative reasoning than they actually were. Thus, the TmL team described problems in the workplaces they studied that were only hypothetically challenging. Had Handel surveyed these same workers, they might also have reported that they rarely solved hard problems.

Outside the workplace, the Decision Research group portrays the kind of problems that people encounter in making everyday decisions about personal health care. These portrayals appear to be based on the researchers' experience or literature reviews and not systematic research. Reyna, Nelson, Han, and Dieckmann (2009) and Hibbard et al. (2007) describe typical problems related to personal health care: having to perform basic arithmetic operations on information embedded in a document, interpreting the information on medication prescription or nutritional labels, choosing among hospitals based on comparative data, and estimating risk magnitudes. While these problems sound less challenging than those facing engineers and midlevel manufacturing and finance workers, they may be more problematic for most people because they arise infrequently and thus never become routine, as do many workplace problems. Indeed, as will be discussed later, most people struggle to solve these health care–related problems.

In sum, we lack a comprehensive view of the kinds of problems people must solve in 21st-century work and life, but generalizations are probably impossible anyway. Some workers solve complex problems that require mathematical interpretation and reasoning, some do not, and some who don't would probably accomplish their jobs better if they did. The advent of sophisticated, mathematized processes and computerized representations of their output may have elevated the need for people to "make sense of mechanism" (Kaput, Noss, & Hoyles, 2008), as is the case for structural engineers and midlevel manufacturing and finance workers, but more research is needed to determine the ways and extent to which this is true and for what segments of the population.

Competencies Required by 21st-Century Work and Life

Researchers have taken many routes to try to determine the competencies required by modern work and life. At the most macroscopic end are studies that examine the impact of schooling in general on national and individual economic outcomes. These studies are valuable in that they measure real outcomes, but they offer only blunt proxies for cognitive competencies and problem-solving success. They are relevant to our discussion only if we presume that "going to school" yields competencies needed in the workplace and that economic gains indicate worker effectiveness in solving workplace problems—presumptions we problematize later.

Very generally speaking, time spent in school correlates with both individual and national economic gains (Barton, 2000; Bowles, Gintis, & Osborne, 2001; Cawley, Heckman, & Vytlacil, 2001; Goldin & Katz, 2008; Heckman, Stixrud, & Urzua, 2006; Tienken, 2008). Unclear, however, is *why* schooling produces economic gains—what schooling imparts that matters (Pellegrino & Hilton, 2012). An obvious explanation—that schooling impacts the economy by increasing cognitive ability—has been tested repeatedly, with conflicting or inconclusive results (Bowles et al., 2001; Cawley et al., 2001, Hanushek & Woessman, 2008, 2011; Murnane, Willett, & Levy, 1995; OECD, 2010). Furthermore, the impact of schooling on national economic outcomes may vary by educational level. The effect is unquestionably strong in developing countries. Elevating a national average from elementary to middle-school education, or elevating low cognitive abilities to moderate ones, yields greater economic gains than adding years of more advanced schooling or high-level cognitive skills (Bowles et al., 2001; Hanushek & Woessmann, 2008; Tienken, 2008). (In fact, Tienken argues that causality runs in the opposite direction in highly educated countries: thriving economies beget high education levels.) The value of college, however, has recently come under scrutiny. College degrees certainly advantage individuals in employment seeking (Greenstone & Looney, 2011; Zaback, Carlson, & Crellin, 2012), but experts differ on how much countries stand to gain by increasing their college-going populations (e.g., Goldin & Katz, 2008; Handel, in press; Lim & Kim, 2013; Vedder, Denhart & Robe, 2013; Wolf, 2009).

Technology appears to mediate the relationship between schooling or cognitive ability and economic outcomes, although, again, there is little agreement about how. Goldin and Katz (2008) contend that rising U.S. income inequality is the combined result of technological advances and a drop in college-degree completion—a low-supply, high-demand situation that elevates the individual economic return on a college degree. Closer inspection, however, reveals a "hollowing out" effect: midlevel white-collar work is disappearing, due, as some argue, to automation, rendering the greatest job growth at the "ends," in areas of low and high cognitive demand (manual labor and cutting-edge innovation) (Barton, 2000; Brynjolfsson & McAfee, 2011; Krugman, 2011). This bodes poorly for increased college education as an overall workforce-development strategy. Globally, technology exerts varied demands on different nations' needs for cognitive ability. To make economic progress, developing countries need only imitate other countries' technologies (an endeavor with lesser cognitive requirements) while developed countries must advance through innovation, which requires greater cognitive skill (Hanushek & Woessmann, 2008, 2011).

Many studies have attempted to trace the economic effects of schooling to specific courses, particularly mathematics, again with conflicting findings. Altonji (1995) found each additional 10th–12th-grade mathematics course predicted a very small earnings benefit for individuals, compared to the benefit of a year of school in general, and no benefit to wage *growth* over the first few years of work. Levine and Zimmerman (1995) documented a somewhat larger wage benefit for high-school mathematics courses, but only for female college graduates. These weak effects for mathematics courses may be a function of averaging a range of high-school courses, possibly diluting stronger earnings returns to more advanced courses with the weaker returns to more basic courses. Interestingly, Bishop and Mane (2004) found that, among high-school courses, career-and-technical-education courses had the biggest earnings impact, an effect that was amplified for students who attended college.

The difficulty of attributing the economic outcomes of schooling to specific courses raises the question: What skills, knowledge, or behaviors are learned in school that matter in the workplace? Various surveys of employers reveal their desire for workers who possess “soft skills” related to collaboration and communication, personal attributes like industriousness and perseverance, and general, higher-order, cognitive skills used in problem solving and critical thinking. Three-hundred Fortune 500 executives surveyed by MetLife (2010) felt the most important areas for college and career readiness were problem-solving skills, critical-thinking skills, clear and persuasive writing, and the ability to work both independently and on teams; they considered higher-level mathematics and science skills far less important. Similarly, 100 U.S. business leaders surveyed by the Business Council (2013) rated the most important skills/capabilities for workers, in order, as work ethic, teamwork, decision making, critical thinking, basic reading and math, and computer literacy. The Partnership for 21st-Century Skills identified four broad areas of employer-desired skills: *core subjects and 21st-century themes; learning and innovation skills*; *information, media, and technology skills*; and *life and career skills* (P21, 2009). Learning and innovation skills subdivided into three categories: *creativity and innovation, critical thinking and problem solving*, and *communication and collaboration*. Relevant to our chapter is their articulation of *critical thinking and problem solving*:

- Reason Effectively
    - Use various types of reasoning (inductive, deductive, etc.) as appropriate to the situation.
- Use Systems Thinking
    - Analyze how parts of a whole interact with each other to produce overall outcomes in complex systems.
- Make Judgments and Decisions
    - Effectively analyze and evaluate evidence, arguments, claims and beliefs.
    - Analyze and evaluate major alternative points of view.
    - Synthesize and make connections between information and arguments.
    - Interpret information and draw conclusions based on the best analysis.
    - Reflect critically on learning experiences and processes.
- Solve Problems
    - Solve different kinds of nonfamiliar problems in both conventional and innovative ways.
    - Identify and ask significant questions that clarify various points of view and lead to better solutions.

Overall, such employer surveys are useful for their “close-to-the-ground” perspective on the requirements of the 21st-century workplace, but they do not empirically link these competencies to actual outcomes such as productivity, employment, or wages. (A 2013 Gallup study of recent graduates, ages 18–35, however, links school experience with these skills to self-reported success in the workplace.) Such surveys may, however, help explain the mechanisms by which schooling contributes to such outcomes. Studies comparing high-school graduates to earners of high-school equivalency-exam certificates suggest that valuable “noncognitive traits” are acquired through attendance in school and rewarded with increased earnings, especially for women and for men in low-skill markets (Heckman et al., 2006). Relatedly, Cawley et al. (2001) found that specific behavioral and social skills impacted earnings independently of cognitive skills, although they seemed to operate by increasing school attendance and performance. Unfortunately, these and other reports (Business Council, 2013; Fischer, 2013; P21, 2006) make clear that many U.S. workers, even those with college degrees, lack the interpersonal and problem-solving skills, and the work ethic, that employers desire.

What of the *mathematical* requirements of modern work and life? In contrast with current policy rhetoric, researchers have observed that most work over the past few decades has involved only basic mathematics. In the 1995 National Job Task Analysis (Packer, 1997), 3,000 U.S. workers across levels reported on the skill requirements of their jobs. Only one of the 25 competencies that they rated most important (Number 14) was mathematical: “perform arithmetic.” Further analysis showed that the content of the most basic high-school algebra and geometry courses more than covered the mathematical skill requirements of the vast majority of workers. Apparently little has changed since the 1990s. Handel (in press) interviewed 2,000 workers across levels and found that, whereas most workers used arithmetic on the job and about 66% used fractions, decimals, and percentages, only about 25% used more advanced mathematics, usually simple algebra. (In an interesting exception, 15% to 30% of “skilled blue-collar workers” used geometry, trigonometry, inferential statistics, and complex algebra—similar to the rates of use among managers and professionals—while all other groups rarely did.) The U.S. Bureau of Labor Statistics confirms these worker self-reports of low mathematics requirements in its 2001–2012 projections for job openings (Barton, 2006). So, too, do the findings of a National Center on Education and the Economy (NCEE) (2013) study of U.S. community colleges. (Because community colleges provide the bulk of U.S. vocational education, their course requirements arguably represent a baseline for career readiness.) For the initial credit-bearing courses in the eight most popular community-college programs, middle-school-level mathematics—especially arithmetic, ratio, proportion, expressions, and simple equations—was most important. Only one program required Algebra 2. Yet many programs required mathematical skills not emphasized in high school: schematics, geometric visualization, complex applications of measurement, mathematical modeling, statistics, and probability. As with soft, social, and general problem-solving skills, many workers and community college students lack even these basic mathematical skills (Murnane & Levy, 1996; NCEE, 2013; Packer, 1997). We have previously noted the shortcoming of such studies: They are constrained by their respondents’ mathematical skills. They cannot reveal whether improved mathematical skill, understanding, or application ability among workers would increase the number of topics and level of mathematics they used and whether that would, in turn, enhance their productivity or work quality.

More or better mathematics learning might also enhance life outcomes. Studies conducted by the Decision Research group showed more numerate people making better health and financial decisions and enjoying better health and financial outcomes. Echoing workplace studies, these authors report that many people lack the requisite numeracy for such decisions, even though the mathematics involved is basic, suggesting that improved mathematical skill would enhance many lives (Peters, Hibbard, Slovic, & Dieckmann, 2007). Their methodology is, in itself, illuminating: *Numeracy* is treated as a somewhat hybrid construct, measured by a test of quantitative reasoning in context. As such, it is theoretically decoupled from schooling, education level, and intelligence (Nelson et al., 2008; Peters & Levine, 2008). That is, the independent variable (numeracy) captures the ability to apply particular mathematical concepts, overlapping somewhat with the dependent variable (making real-world decisions involving quantity). This might shed some light on why years of high-school mathematics—a purer measure of mathematical competence—poorly predicts workplace performance (i.e., earnings): The mathematics in high-school courses is probably more topically advanced than necessary for real-world decisions, but such courses do not enable students to apply mathematics in real-world problem solving. As Pellegrino and Hilton (2012) note, "Over a century of research on transfer has yielded little evidence that teaching can develop general cognitive competencies that are transferable to any new discipline, problem, or context, in or out of school" (p. 8).

How and Where Might 21st-Century Competencies Be Developed?

Despite their pessimism about teaching for transfer, Pellegrino and Hilton (2012) conclude their review of 21st-century competencies with a call for exactly that. Indeed, for them, what makes something a 21st-century competence is that a person can apply it in situations different from the one in which it was learned. Drawing on a large body of education research, they propose that schooling should aim for "deep learning"—an understanding of the general principles or structures that underlie concepts and problems—because deep learning promotes transfer.

Pellegrino and Hilton concede uncertainty about what schooling imparts that promotes positive outcomes in 21st-century work and life, and even about whether the current labor market truly demands increased cognitive competence. Regardless of this uncertainty, or maybe due to it, they put their faith in school-based improvements to promote adult success. Specifically, they recommend transforming teaching to focus on:

- underlying principles
- the process of learning
- high-level skills, even if low-level skills have not yet been mastered
- learning skills and concepts in a specific domain
- the conditions for application
- using multiple representations, especially graphic or tactile
- integrating noncognitive skills
- giving the learning process ample time.

These recommendations are presumed to increase the likelihood of transfer and produce more effective problem solvers in an unpredictable and fast-changing workplace or society because the knowledge and skills learned can be employed flexibly and across a range of problem situations. Still, Pellegrino and Hilton acknowledge the power of context, recommending teaching the use of skills and knowledge in specific domains, making explicit the conditions in which the knowledge might be applied, and assessing its application. As such, Pellegrino and Hilton's recommendations (albeit not aimed at mathematics specifically) depart from a view that prevailed for decades: that mathematics education provides general mental training/discipline that can transfer across fields (Stanic & Kilpatrick, 1989).

Responding to the low level of mathematical topics required in profession-oriented community-college courses and students' inability to apply even this low-level mathematics, the NCEE's (2013) recommendations for mathematics education generally echo Pellegrino and Hilton's. The NCEE urges K–12 schools to spend far more instructional time on proportional relationships, percents, graphical representations, functions, expressions, and equations, emphasizing conceptual understandings of these topics and their application to practical problems. Contradicting current policy statements, but reflecting Pellegrino and Hilton's appeal for ample time to the learning process, the NCEE argues against requiring Algebra 2 in high school and advises delaying Algebra 1 for some students. Further, the NCEE advocates for multiple mathematical paths, not just the traditional one leading to calculus. Options in statistics, data analysis, and applied geometry, for example, would better reflect the mathematics used across occupations and retain more students in the STEM pipeline.

Other scholars (e.g., Bishop, 1993; Fischer, 2013; Harvard Graduate School of Education [GSE], 2011) argue for contextualized or job-specific learning, either in school or in workplace settings. Noting differences among the qualities that constitute "readiness" for the workplace, college, and healthy personal development, respectively, the *Pathways to Prosperity* report (Harvard GSE, 2011) concludes that "a more holistic approach to education—one that aims to equip young adults with a broader range of skills—is more likely to produce youth who will succeed in the 21st century" (p. 4). This more holistic education includes vocational education, career counseling from the early grades, and structured workplace experiences in high school. The report lauds work-related K–12 educational experiences such as the engineering curriculum, "Project Lead the Way", California's "Linked Learning" initiative, robotics competitions, and various career-and-technical and career-academy programs. And while over 90% of the U.S. CEOs surveyed by the Business Council (2013) rated secondary and four-year college education "very/most important" for a top-quality workforce, 82% felt the same way about "on-the-job training." The TmL researchers aimed their interventions at experienced workers but proposed that gaps in adult TmL might be lessened by school instruction that acknowledged the importance of context, real-world constraints, action, and responsibility (Bakker et al., 2008). Empirical support for this idea comes from the Gallup (2013) study showing that school experience with real-world problem solving, more than with any other 21st-century skill, predicted self-reported success at work.

The potential contributions of work-based or work-related learning to 21st-century problem solving are twofold: First, it provides realistic contexts for learning that could help overcome the failure of abstract, general knowledge to transfer and thus promote the use of academic knowledge for solving real problems. (Less clear, however, is whether context-specific learning enables knowledge to be used in solving problems beyond the context in which it was learned [Cognition & Technology Group at Vanderbilt, 1990].) Second, work-related learning is likelier to engender the soft skills and attitudes desired by employers, especially if it involves work on projects with real purposes and clients. But blending work (or real-world) situations and classroom learning is not a straightforward matter. The TmL researchers warn that, "Formulating potential implications of workplace research for school education is a tricky business…. One should not make the mistake to try and copy workplace situations in school education" (Bakker et al., 2008, p. 142). Indeed, a current topic of contention is whether 21st-century competence is better learned in school or in workplace or other out-of-school settings (Fischer, 2013). Fifty-nine percent of the young employees in the Gallup (2013) study reported learning most of the skills needed in their current job outside of school, but this percent dropped significantly for college graduates. The implications of these findings are unclear: they may suggest room for improvement in the way high schools prepare students for the workplace, or they may indicate that the kinds of jobs filled by high-school graduates rely less on intellectual, school-taught skills than do jobs requiring a college degree.

Advancing Problem Solving in the Mathematics Curriculum

Recent research about 21st-century work and life, summarized in the prior section, yields conflicting perspectives about how technology has impacted workplace problem solving and whether cognitive demand is rising in general. Nevertheless, some principles emerge from this research with direct relevance to mathematics education:

- Problem solving in work and life requires a more solid and flexible grasp of basic mathematics than much of the population currently possesses. Advanced mathematics courses do not appear to be the solution.
- Certain noncognitive and general skills (that are typically underpromoted in education) are critical for workplace problem solving. Many of these are cognitively high level.
- Many jobs, particularly in IT-intensive fields, require an understanding of conceptual models that underlie processes or systems ("making sense of mechanism"), which in turn requires interpretations of complex representations within the work context and a deep understanding of the work domain.
- In some contrast, everyday life decisions increasingly require interpreting quantitative data in various complex forms, in multiple, unfamiliar domains.
- The ability to apply one's training and knowledge to novel, unfamiliar problems (transfer) is highly privileged by employers, and is presumed most effectively fostered when learning occurs in work-based contexts on the job or replicated in schools.

In light of these principles, we revisit the debates presented earlier and consider how we might approach mathematical problem solving within the school curriculum for the purpose of preparing students for success in 21st-century work and life.

Problem Solving: Process and Content

Earlier, we described a decades-long debate on teaching problem solving versus teaching mathematics *through* problem solving. Mathematical-content knowledge per se is almost never an explicit goal of employers, but their clear desire for workers who are effective problem solvers implicitly argues for schools to teach problem solving as an end in itself. Thus, despite the mathematics-education community's recent bent towards problem solving as a vehicle for learning mathematical content, we recommend honoring both goals. We advocate restructuring this debate to ask, instead, how we might design problems that are sufficiently cognitively demanding to foster both significant mathematical content and effective problem-solving capabilities. As a start, we might consider redefining problem solving as an experience where the solver or a collaborating group "needs to develop a more productive mathematical way of thinking about the given situation" (Lesh & Zawojewski, 2007, p. 782). The focus then becomes one of learning or idea generation, rather than the application of problem-solving processes or strategies. Hence, one key feature of problem solving that promotes both process and content is the opportunity for student generation of mathematical ideas, indeed, even before such content is formally introduced.

This feature of idea generation reflects calls for more cognitively challenging tasks that encourage high-level thinking and reasoning, have multiple points of entry, and enable the use of varied solution approaches. Unfortunately, as Silver, Mesa, Morris, Star, and Benken (2009) report, emphasis in the 1990s on the importance of cognitively demanding tasks (e.g., Stein, Grover, & Henningsen, 1996) appears to have gone largely unheeded. This is a pity because Stein et al.'s criteria provide a pertinent basis for designing mathematics-curriculum problems that target 21st-century demands related to communication and other general problem-solving skills. For example, problems with high cognitive demand require students to explain, describe, and justify; make decisions, choices, and plans; formulate questions; apply existing knowledge and create new ideas; and represent their understanding in multiple formats. Students are likely to face such demands when encountering problems outside of school, where uncertainty and a broadening of mathematical content call for problem solvers who have the disposition and ability to generate mathematical knowledge on an "as-needed basis." Although debates continue about whether the cognitive challenges of the workplace are truly increasing, we do know that more skillful decision making and problem solving are needed in all avenues of life, where solving information-laden problems has become increasingly vital to one's overall health, well-being, and achievements.

General Skills and Heuristics

Collectively, the general skills for successful problem solving advocated by employer groups and mathematics educators share some features, although recommendations for fostering these skills remain challenging and, at times, contradictory. From the extensive literature on general skills, the four broad areas of employer-desired skills that have been identified by the Partnership for 21st-Century Skills (P21, 2010) appear particularly pertinent. These areas, which we have delineated in a prior section, include effective reasoning, using systems thinking, making judgements and decisions, and solving problems. These skill areas are reflected in recent writings of Schoenfeld (2011, 2013) and Lester (2013), who raise the importance of recognizing and constructing patterns of inference and making careful judgments during the problem-solving process. Schoenfeld's (2013) inclusion of solvers' beliefs and dispositions about themselves and the discipline being engaged, together with their "decision-making mechanism" (p. 17), is an interesting extension of his earlier work, in which "dispositions, beliefs, values, tastes, and preferences" (Schoenfeld, 2010, p. 29) were identified as core features of successful problem solving. More recently, Swanson (2013) also highlighted the importance of students' beliefs and orientations when faced with challenging mathematics problems. She aptly titled her article "Overcoming the Run Response," invoking the fear such problems can instill in students. In revealing students' emotional reactions to these problems, Swanson stressed the importance of self-awareness and regulation, in other words, metacognition. Metacognition, in broad terms, is increasingly recognized as playing a critical role in successful problem solving, both within and beyond the curriculum (e.g., Lester, 2013; Pellegrino & Hilton, 2012). Lester's (2013) inclusion of "intuition" as one of the key components of successful problem solving further supports the recognition of general cognitive components, provided students become conscious of their intuitions and can evaluate their implications for the problem at hand. Yet, as he and others (e.g., Schneider &

Artelt, 2010) have lamented, we still know little about how to develop students' metacognitive abilities. Indeed, current approaches may be inherently self-contradictory (see Kirshner, Chapter 4 in this volume).

The extent to which these competencies and dispositions exist as general abilities that individuals can possess and apply across domains, much less be taught, remains unclear. The strong consensus on their importance in the workplace could mean employers have actually seen workers—their most valuable ones—repeatedly solve nonroutine problems, implying that such general competencies exist. Even so, *nonroutine* should not be confused with *contextually unfamiliar*, a broader condition. Exemplary workers have the ability to solve novel problems within their domain (e.g., auto mechanics) but there is no reason to expect this ability to extend to problems in other domains (e.g., cooking). This distinction might inform the issue of problem-solving heuristics: within-domain heuristics (e.g., for mathematical problem solving, as Polya's were originally intended) might hold promise, but not cross-domain heuristics. Still, even the observation of able solvers of nonroutine problems within a domain does not guarantee that within-domain heuristics explain their success or even exist. Clearly, more research is needed on within-domain problem-solving expertise and how it is learned.

Contexts and Authenticity

It is apparent that domain knowledge is key to workplace problem solving. Beyond specific job training, however, it is hard to see how to address this in schools. Further, while it may be possible to succeed at work with expertise in only a single domain, everyday life decisions occur in multiple domains encountered too infrequently to become familiar (e.g., medical and financial decisions). Still, with little research support for the transfer of mathematical skills learned in the abstract to contextual problems, it seems advisable to engage students in learning to solve contextualized problems with mathematical tools and in making sense of models (real-world systems and processes that mathematically relate quantities). As more classrooms employ context-based mathematical learning and modeling, more research can be conducted on the effect of such education on students' problem-solving ability in future courses, work, and life—specifically, how well learning to use mathematics to solve problems in certain contexts (in school) prepares students for doing so in other contexts outside of school.

Even if such transfer remains hard to prove, there are other reasons to teach mathematics in the context of solving realistic problems. It offers a more realistic view of real-world problem solving, which is usually interdisciplinary and dependent on contextual specifics (unlike traditional school math problems); this view should better prepare students for such problem solving. It could also reveal the nature of various workplaces, thus building career awareness and a general understanding of what adult work requires. Realistic problem solving could also invoke and build noncognitive skills desired by employers, such as collaboration and communication. It might also build positive dispositions towards the use of mathematics in solving real-world problems, convincing students that mathematical tools can be useful in problem solving, developing self-efficacy in the use of those tools, and showing that persistence is often required.

Mathematical Content

Here, the message from the research is clear: For solving the vast majority of problems arising in work and life, only basic mathematics is needed, but people need to be far more fluent with its use and application than they are today. The current press to expose more students to more advanced mathematics topics before college appears to head in the wrong direction, especially when these topics are covered rapidly and cursorily. More advisable would be spending more time with each topic, enriching students' understanding by using the topic to solve a variety of mathematical and real problems. For most students, a goal of mastery of basic algebra and geometry by the end of high school seems most justified, with the addition of the less-traditional topics of statistics, data analysis, and solid geometry.

Despite the shortcomings of transfer research, the finding by the TmL researchers of the importance of understanding the conceptual models underlying real-world processes supports the idea that deep understanding of somewhat generalizable concepts is more efficacious in promoting problem-solving ability, at least within a domain, than shallower, situation-specific, procedural knowledge. This suggests that whatever the mathematical content (and whatever the context), the primary goal for teaching should be deep understanding of the underlying principles and concepts. This should work in both directions: problem solving in real contexts can contribute to a deep understanding of mathematical concepts and principles, and applying mathematical tools to real problems can contribute to a deep understanding of the concepts and principles underlying the real-world systems or phenomena. The sentiments of employers and the observations by workplace ethnographers underscore the importance of learning and metacognition on the job: successful engineers, scientists, and technology workers use mathematics or quantitative reasoning to better understand the systems at the heart of their work, at the same time honing their mathematical or quantitative "tools" for future problem solving. Schools should make explicit to students that this learning cycle is part of what the best STEM workers do.

Mathematical Modeling—One Way to Prepare for 21st-Century Demands

Mathematical modeling is becoming increasingly important in the workplace and in many other avenues in life. The terms *models* and *modeling* have been interpreted variously in the literature (e.g., English, 2013; Gainsburg, 2006; Lesh & Doerr, 2003; Lesh & Zawojewski, 2007; Stillman & Galbraith, 2011), with debate over whether these are components of the broad problem-solving spectrum or entities in their own right. Without taking a stance on this debate, we consider modeling a powerful vehicle for bringing features of 21st-century problems into the mathematics classroom.

Modeling has an extensive history within the mathematics community, as can be seen in the 1983 establishment of International Conferences on the Teaching of Mathematical Modeling and Applications (ICTMA; Kaiser, 2010). Various interpretations and forms of models and modeling exist, even within the ICTMA community, but we refer to Lesh and Fennewald's (2010) basic "first-iteration definition of a model," namely, "A model is a system for describing (or explaining, or designing) another system(s) for some clearly specified purpose" (p. 7). This interpretation is especially germane to fields beyond mathematics education, including engineering and other mature science domains. Some of the experiences that engage students in modeling from this perspective have been described as model-eliciting activities (MEAs; English, 2007; Hamilton, Lesh, Lester, & Brilleslyper, 2008; Lesh & Doerr, 2003), where the focus is on the processes of interpretation and reinterpretation of problematic information, and on the iterative development of mathematical ideas as models are formed, tested, and refined in response to certain specifications. MEAs give students the opportunity to create, apply, and adapt mathematical and scientific concepts in interpreting, explaining, and predicting the behavior of real-world based problems such as those that occur in engineering (e.g., Gainsburg, 2006).

Exposure to statistical information in these MEAs provides a valuable basis for developing the skills that consumers need in working effectively with data. Interpreting and understanding the implications of insurance documents, financial agreements, and political agendas, to name a few, requires an ability to deal with complex information. Modeling problems present students with such data, which must be interpreted, differentiated, prioritized, and coordinated to produce a solution model. Furthermore, students' modeling work often elicits nontraditional mathematics topics for their grade level, because different types of quantities and operations are needed to deal with realistic situations. For example, MEAs often involve accumulations, probabilities, frequencies, and ranks, with the associated operations of sorting, organizing, selecting, quantifying, weighting, and transforming large data sets (Doerr & English, 2003; Lesh, Zawojewski, & Carmona, 2003). Integral to the mathematizing process are the myriad representational media required in expressing and documenting the models, including computer-based graphics, tables, lists, paper-based diagrams and graphs, and oral and written communication (Lesh & Harel, 2003). Because these representations embody the factors, relationships, and operations that students considered important in creating their models, MEAs offer an additional benefit to teachers and researchers: powerful insight into the growth of students' mathematical thinking.

As we noted in our discussion on 21st-century demands, the importance of understanding the underlying models that are represented mathematically and technologically is crucial in many fields, including engineering, finance, manufacturing, and agriculture. Virtually all aspects of modern life have been mathematized using the modeling components we have highlighted; our future citizens need to be aware of, and understand how, this mathematization shapes their lives in so many ways. In preparing our students to become mathematically aware, consideration needs to be given to how we might select contexts that approximate authenticity and foster an appreciation of learning through classroom problem solving. Modeling problems also foster the types of general skills that employers demand in the workplace and that citizens need for maximum societal participation. Such skills include critical and innovative thinking, complex reasoning, metacognitive actions, and collaboration and communication within and across disciplines. In sum, modeling activities represent an excellent example of an instructional strategy that should promote the kind of learning valued in 21st-century work and life.

Concluding Points

We commenced this chapter by reviewing some of the key debates on mathematical problem solving over past decades. Given that these debates remain largely unresolved, we turned to research on the demands of modern work and life and examined drivers of change in these settings. We considered the nature of the problems that need to be solved in these changing contexts and the competencies required for dealing effectively with the challenges that arise. In light of this research, we revisited mathematics education and suggested how we might better prepare students for successful problem solving in the 21st century.

Where do we stand then, with regard to suggestions for how we might advance students' problem solving in today's world? Clearly, there are many courses of action that might be adopted and, with the diverse range of learning contexts we face, no one set of recommendations would suit all school systems. Nevertheless, we have focused on ways in which we might teach problem-solving processes in conjunction with developing mathematical content, how we might address context and authenticity, and how general skills might be developed to enhance problem-solving success in the modern workplace and life. Many challenges remain, however, in implementing our recommendations and issues for further investigation abound. We address just a couple of such issues in closing.

We have given MEAs as an example of a rich source for developing both problem-solving processes and mathematical content as well as providing authentic contexts and fostering general skills. Other cognitively demanding problem types that offer similar learning opportunities should also be incorporated within the mathematics curriculum. Interdisciplinary problems that require synthesizing knowledge from across STEM domains—for example, problems in an engineering context—can be appealing and authentic, but they remain rare in the mathematics curriculum (English, Hudson, & Dawes, 2012; Suh, Seshaiyer, Moore, Green, Jewell, & Rice, 2013).

Modifying "traditional" mathematical problems to foster idea generation rather than mere procedural application is another area in need of attention. Such modification can be both teacher and student initiated. Teachers can restructure existing problems to be interdisciplinary (perhaps in collaboration with teachers of other subjects), so that obvious solution paths become less apparent, not all of the required mathematics is presented, and a more open approach to solution is encouraged. And the renewed interest in problem posing offers valuable suggestions for engaging students in adapting, creating, and solving their own problems, beginning with such experiences in the earliest years of schooling. For example, English and Watson (2014) have shown how problem posing can be integrated within the regular mathematics curriculum in the areas of statistics and probability, where students direct their own investigations.

Thinking more broadly, we advocate an increased awareness of mathematical problem solving beyond the classroom and greater insights into how the demands of the 21st century are impacting our lives. While we cannot simply transport the mathematical problems of the outside world into the classroom, there are many ways in which we can more realistically contextualize "school problems." Such recontextualization should incorporate a transition to more cognitively challenging problems—ones which stimulate curiosity, foster critical thinking, promote creative solutions, and feature multiple entry and exit points for increased access by a range of students.

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