



Title of Research: A Study of Teaching the Skills of the Primary School Mathematics Curriculum: Teaching Equivalence in context

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Brief outline of research idea:

Equivalence is a rich mathematical idea that can “unify the various subjects and topics” of the primary mathematics curriculum (Artzt & Newman, 1991, p. 128). Student mastery of equivalence provides a foundation for subsequent success in algebra courses (e.g. Falkner, Levi, & Carpenter, 1999; Rittle-Johnson, Matthews, Taylor, & McEldoon, 2011). Several articles have documented challenges experienced by children in acquiring concepts of equivalence (e.g. McNeil, 2008) and some have proposed levels of knowledge of equivalence held by primary school children (Rittle-Johnson et al., 2011). Although knowing about the challenges experienced by children in learning equivalence and being aware of possible learning sequences may help teachers to *plan* their teaching, the complex nature of classroom instruction means that implementing such a plan may be problematic (Lampert, 2001). This study describes one teacher’s attempt to apply research findings on learning equivalence in one classroom situation. Furthermore, in planning the classroom work, the teacher extended the meaning of equivalence found in many mathematics education articles beyond using the term as a synonym for equality (e.g. Rittle-Johnson et al., 2011). The definition of equivalence used is of a relation that satisfies the properties of reflexivity, symmetry and transitivity (Daintith & Clark, 1999) and where equality is one example of a relationship that satisfies such properties.

1. Summary of research aims

This report describes and analyses one excerpt from a series of lessons designed and delivered to promote understanding of the mathematical concept of equivalence among third class students in Ireland. The research question that guides the study is “How were the teacher’s knowledge, beliefs and goals evident in the decisions made during a series of lessons focused on equivalence taught over a period of ten hours?” The question will be answered by analysing interactions among the third-class students, the teacher and the mathematics in a ten-hour mathematics summer school. The purpose

of answering this question is to generate insights into the complexity of planning and teaching a unit focused on mathematical equivalence. The findings will be helpful to those who offer professional development in mathematics to teachers and to those who study the practice of teaching.

2. Summary of background reading

Three perspectives inform this study: research on teaching equivalence, research on teaching in context, and studying one's own practice. Mathematical equivalence is a relation defined on a set that satisfies three properties: reflexivity, symmetry and transitivity (e.g. Daintith & Clark, 1999). Because equivalence is a relation, understanding it is important in helping students make the transition from computing to relational thinking, an important step on the way to learning algebra (Baroody & Ginsburg, 1983; Carpenter, Franke, & Levi, 2003). Furthermore, learning the properties of equivalence – reflexivity, symmetry and transitivity – is important in itself. For example, Baroody and Ginsburg (1983) claim that if the equals sign is introduced in the context of the reflexive property, students may be less likely to view the sign (exclusively) as an operator. Furthermore, understanding the transitive property appears to be a requirement for using unit iteration when measuring length (Kamii, 2006). Many research articles call for more emphasis on the topic of equivalence, especially around the meaning of the equals sign (e.g. McNeil, 2008; Rittle-Johnson & Alibali, 1999). In the Irish mathematics curriculum the term equivalence is mentioned frequently in the learning outcomes for junior classes; students are expected to be able to “compare equivalent and non-equivalent sets” (Government of Ireland, 1999, p. 22). But the term is generally used as a synonym for equality, with the clarification that “equals means ‘the same’ or equivalent” (p. 23, 45). Although equality is an example of an equivalent relation, other relations (e.g. “is parallel to”) are also equivalent. In the Irish mathematics curriculum for senior classes, equivalence is mentioned almost exclusively in the context of fractions, decimals and percentages – although currency equivalents are mentioned in relation to money (Government of Ireland, 1999). The properties of equivalent relations – symmetry, reflexivity and transitivity – are not mentioned. Consequently, if teachers faithfully implement the written curriculum, students could acquire a narrow conception of equivalence. This study describes an attempt to build students' understanding of equivalence as a mathematical relation that possesses particular properties.

In the practice of teaching a teacher makes many decisions, many of them simultaneously (Lampert, 2001). This is particularly the case where a teacher wishes to promote classroom discourse and to elicit children's thinking. For example, a teacher decides what tasks to use, what questions to ask, who to invite to respond to the question, how long to wait for a response, which ideas to pursue and which

to drop, and whether to work with individuals, pairs of children or the whole class. Some decisions may become automatic for experienced teachers (Schoenfeld, 2011), in the form of routines (Leinhardt & Greeno, 1986; Leinhardt & Steele, 2005), for example. However, not all decisions are automatic and even those that become automatic with experience, must be made by novice teachers (Lampert, Beasley, Ghouseini, Kazemi, & Franke, 2010), so better understanding of the decisions teachers make is desirable. Schoenfeld (2011) offers a framework which can be used to study teacher decision making.

Schoenfeld's (2011) model proposes that a teacher's actions can be explained by the teacher's decisions and the teacher's decisions can be explained by the teacher's orientations, goals and resources. Schoenfeld uses this model to analyse video representations of teaching and to describe how a teacher's resources, goals and orientations shapes the teacher's actions. He uses the data obtained from analysing the teaching to build a detailed model of the teaching. This is done by means of iterative parsing of the record of the teaching. This study uses Schoenfeld's approach to parse one segment of teaching from the unit taught on equivalence.

The study is an example of practitioner-inquiry research (Cochran-Smith & Donnell, 2006). As both author of the study and a subject of it, the author is a registered primary school teacher who works fulltime as a teacher educator. The kind of teaching to which he aspired was "teaching that aims to take seriously both mathematics as a discipline and children's mathematical ideas, and that sees mathematics as a collective intellectual endeavour situated within community" (Ball, 1999, p. 29). In order to teach in this way, students were encouraged to engage in mathematical practices such as reasoning about their answers, communicating and expressing their thinking and representing their mathematical ideas.

3. Outline of methodology used including details of how any ethical considerations were addressed

The study analyses in detail one excerpt from a ten-lesson unit of teaching as it was planned and implemented in order to identify how the teacher's goals, orientations and resources were evident in decisions and subsequent actions. Schoenfeld's (1998) model of "teaching-in-context" is used to analyse the teaching. This model acknowledges the complexity of teaching in a particular context and requires explication of the teacher beliefs, lesson goals, and knowledge that informed the teacher's actions and decisions. Transcripts of the lessons have been created. Analysis of the transcripts began with a parsing of selected instances of instruction into what Schoenfeld (1998) calls "action sequences" and further iterative decomposition of "chunks" of instruction "each of which coheres on

phenomenological grounds” (p. 30). The model was used to create a representation of the lesson by identifying action sequences, of continuously reducing grain sizes, which occurred in the teaching unit. The sequences were scrutinized in light of the written lesson plans and goals, the teacher’s beliefs about teaching mathematics, and the teacher’s knowledge base. Schoenfeld (1998) describes this way of analysing each action sequence as follows:

we will want to know whether it was expected or unexpected, how (if at all) it corresponded to the teacher’s lesson image, what “triggered” it, what beliefs might have shaped the way it took place, what goals the teacher’s actions were intended to satisfy, what kinds of knowledge the teacher depended on in the interaction, and what brought the episode to a close” (1998, p. 31)

Such detailed analysis of the instruction made possible a mathematical and pedagogical evaluation of the lesson unit based on a theoretical framework that views equivalence as an area rich in potential for children’s learning of mathematics. Specifically the study investigated how the teacher’s goals, orientations and resources shaped the teacher’s actions across the sequence of lessons. By grounding the analysis in the knowledge resources, beliefs and goals of one teacher, the study can inform how other teachers might design and implement teaching units on equivalence, or other topics, for primary school students. The approach used has the added benefit of considering the effects of both knowledge and beliefs on the teacher’s actions whereas many studies investigate the impact of either one or the other on what teachers do (Bray, 2011).

The analysis of instruction benefits practitioners because it makes available records and analyses of teaching practice which can be discussed and considered in light of other teachers’ own practice. One study found that although pre-service teachers could identify the potential of tasks to develop children’s relational thinking, they had to be convinced of the importance of using such tasks in developing children’s mathematical thinking (Stephens, 2006). The current analysis benefits researchers by providing a practitioner’s perspective on an important mathematics education topic that has frequently been studied in clinical settings, with little, if any, description of how the topic was taught (e.g. McNeil, 2007; Rittle-Johnson et al., 2011).

Students’ participation in the summer maths laboratory school was voluntary because it provided optional, additional tuition for them. Parents who were interested in having their child participate in the laboratory school were invited to an information evening for parents in June 2009. Children were also welcome to attend this meeting. Parents of children were asked to sign consent letters for live observation of the class by teachers, for the collection and use of photographic, video, audio and written records to be used for research purposes and for video, audio, and written records to be used

for educational purposes. Separate consent from parents was sought for all three uses. In 2009 it was not standard practice to seek student assent (although this has since changed) and thus such consent was not sought. The study was approved by the Institute's Ethics in Research Committee.

Data sources, evidence, objects or materials

This study draws on data collected during a ten-hour program – two hours per day over five days – held in the first week of July 2009. The program was designed as (a) a mathematics summer school for third class students (b) a professional development course for primary school teachers and (c) a site for research. The school was set up as an primary mathematics laboratory.¹ As both teacher and researcher the author taught the 25 third-class students (14 boys and 11 girls) while being observed by 26 primary teachers.

The students were a relatively diverse group. They came from ten different schools. At least four of the schools were designated as disadvantaged. Three of the pupils are native English speakers who attend an Irish-medium school so learning mathematics through the medium of the English language on the summer school was new to them. One student whose parents are African attended the class.²

The data collected consisted of good quality audio and video records of the mathematics teaching, transcripts of the audio, the teacher's lesson plans, the posters used in the lessons, and the students' written class work, quizzes, and homework. The multiple types of data collection procedures allow for what Johnson (1997) calls "methods triangulation" as a means of improving the internal validity of the study (p. 288).

Initially for this report three excerpts were chosen from the ten hours of teaching. Two excerpts (one from day 2 and one from day 3) focused on whole-class discussions of whether several number sentences were true or false. In the third excerpt (from the fifth and final day) the children evaluated a series of examples of mathematical relations for their equivalence. Details of the three excerpts are summarised in Table 1. The excerpts were chosen because they were explicitly focused on equivalence-related topics, they each provided examples of whole-class discussions and they occurred at different stages throughout the ten hours of teaching.

¹ Based on a similar idea at the University of Michigan. See http://www-personal.umich.edu/~dball/presentations/041007_AERA_labclass.pdf. The study was funded by the Teaching Council.

² Diversity of students by national origin in Irish schools has been a feature since around 2000. One student represents 4% of the class. About 3% of children in Irish primary schools have African nationality.

Table 1

Summary Details of lesson excerpts analysed

Excerpt Number	Day	Duration	Content
1	2 (second hour)	20' 50"	Class discussion on whether the following number sentences are true or false: $12 - 9 = 3$; $34 - 19 = 15$; $43 = 27 + 16$; $5 + 7 = 11$; $58 + 76 = 354$; $27 + 48 - 48 = 27$; $345 + 568 - 568 = 353$; $48 + 63 - 62 = 49$.
2	3 (first hour)		Class discussion on whether the following number sentences are true or false: $58 + 76 = 2940$; $58 + 76 = 1214$; $58 + 76 = 94$; $58 + 76 = 874$; $58 + 76 = 58 + 76 - 76$; $58 + 76 = 60 + 80$; $58 + 76 = 60 + 74$;
3	5 (second hour)		Testing the following examples of relations for equivalence, by checking if they have the properties of reflexivity, symmetry and transitivity: $50 = 50$; $7 \times 8 = 56$; $10 > 5$; $56 < 60$; a large circle is similar to a small circle; $4 \neq 2$.

The number sentences that children were asked to label as true or false were chosen based on literature about equivalence (see Table 2) and on the needs of the class based on the author's assessment of what would help the children develop the concept of equivalence. Other number sentences were evaluated over the course of the week but they are not part of this analysis.

Table 2

Details of Number Sentences the Students Were Asked to Label as True or False. Sentences 1 to 8 relate to Excerpt 1.

Number	True/false Number Sentences	Notable Feature	Source
1	$12 - 9 = 3$	"Operations-equals-answer" structure (Rittle-Johnson et al., 2011). True.	Carpenter, Franke and Levi (2003)
2	$34 - 19 = 15$	"Operations-equals-answer" structure (Rittle-Johnson et al., 2011). True.	Carpenter, Franke and Levi (2003)
3	$43 = 27 + 16$	Operation on right of equals sign. True.	

4	$5 + 7 = 11$	“Operations-equals-answer” structure (Rittle-Johnson et al., 2011). False.	Carpenter, Franke and Levi (2003)
5	$58 + 76 = 354$	Operating on the numbers not necessary. False.	Carpenter, Franke and Levi (2003)
6	$27 + 48 - 48 = 27$	Operating on the numbers not necessary. True.	Carpenter, Franke and Levi (2003)
7	$345 + 568 - 568 = 353$	Operating on the numbers not necessary. False.	Carpenter, Franke and Levi (2003)
8	$48 + 63 - 62 = 49$	Compensatory strategy (Rittle-Johnson et al., 2011) can be used. True.	Carpenter, Franke and Levi (2003)
9	$58 + 76 = 2940$	Operating on the numbers not necessary. False.	
10	$58 + 76 = 1214$	Operating on the numbers not necessary. False.	
11	$58 + 76 = 94$	Operating on the numbers not necessary. False.	
12	$58 + 76 = 874$	Operating on the numbers not necessary. False.	
13	$58 + 76 = 58 + 76 - 76$	Operations on both sides of the equals sign. False.	
14	$58 + 76 = 60 + 80$	Operations on both sides of the equals sign. Compensatory strategy (Rittle-Johnson et al., 2011) can be used. False.	
15	$58 + 76 = 60 + 74$	Operations on both sides of the equals sign. Compensatory strategy (Rittle-Johnson et al., 2011) can be used. True.	

4. Overview of research findings and recommendations

In this section I analyse the teaching observed in the chosen excerpt. It occurred on the fourth hour of the ten lesson unit and it involved a whole-class discussion of whether eight mathematics sentences were true or false. A pattern can be identified in how the teacher presented each number sentence to the class and discussed. The sentence was displayed on the board, a student was called on to state whether the sentence was true or false, a justification for the response was sought and given, in some cases alternative opinions were sought, and the exchange generally concluded with a brief recap by the teacher. In the excerpt most of the exchanges followed this pattern. When three of the eight number sentences were being discussed, the pattern outlined above was broken. In one instance the teacher asked a student to describe a procedure used to add numbers in his head; second, in response to a question from the teacher a pupil outlined how place value principles could be used to add two two-digit numbers; and third, in response to an incorrect response the teacher tried to support a student in learning that $x - x = 0$. Both in the routine parts of the exchanges and in the non-routine parts, the teacher's goals, orientations and knowledge resources were evident.

Each of the three deviations from the general pattern are of interest because each one, whether initiated by a student or by the teacher, moved the class away from the key focus of the lesson which was to determine if number sentences were true or false and to justify the decisions. A teacher could conceivably omit or cut short these diversions because they are peripheral to the key goals of the lesson in the first two instances, and in the third instance may be an idea that only one student needs to learn. Nevertheless, each one could also be seen to serve broader mathematical goals in teaching children who have just completed third class.

In the first instance, a pupil had justified a claim that $43 = 27 + 16$ is true "because twenty-seven plus sixteen is forty-three." Rather than immediately looking for agreement, a dissenting voice, or an alternative justification the teacher asked "How did you figure out in your head that twenty-seven and sixteen is forty-three?" Such a question seems unusual because the lesson focus is on the justification of the student's answer rather than on a calculating procedure and because the procedure had been calculated correctly. The student's description of the procedure was routine in that even though he had done the calculation mentally, he described it as if he had written it using the traditional two-column addition algorithm: "I added seven and six in my head and took the one plus two and the one." Although other procedures would have been possible, no further procedures were elicited. The student's justification of his claim that the sentence was true provided an opportunity for him to

describe the procedure used for the rest of the class, an opportunity which the teacher used, but it served a broader mathematical teaching agenda than the specific algebra focus in this lesson.

The second instance was also one where the teacher played a role, albeit a less direct one, in moving the focus away from the justification. Two students had given two different, valid justifications as to why $58 + 76 = 354$ is false. When a third student successfully refined the second student's justification of the sentence's falsity, instead of highlighting the refinement the teacher asked another student to repeat what the second student had said. The teacher may have done this because he didn't recognise the third student's justification as being a refinement of the second student's utterance. Nevertheless, the effect of returning to the second student's justification was to move the discussion to the topic of place value and adding five tens and seven tens, and away from the discussion of equivalence. This move again seems like an attempt to address a broader idea in mathematics (place value) than the lesson focus on algebra.

The third deviation from the main goal of the lesson was different. The teacher asked a student to justify whether the sentence $345 + 568 - 568 = 353$ was true or false. Despite the previous number sentence being based on a similar idea, the student who was asked to respond did not seem to understand the idea that $568 - 568 = 0$ and her initial claim was that the sentence was true. Although the idea that $x - x = 0$ is important for the lesson, it is possible that the difficulty was one held only by this student. The teacher, however, pursued the idea in the lesson rather than, or as well as, following up with the student one-to-one at another time. One reason for doing this may be because the teacher wanted other students to strengthen their understanding of the idea by asking them to help explain the idea to the student. This may also have been done because the teacher knew little about the students' prior knowledge and consequently would not have known if other children had the same difficulty.

These deviations suggest that in addition to the broad lesson focus, the teacher held other, possibly implicit, goals for the lesson. He may have decided to pursue potentially productive mathematical paths if the opportunity arose. Such opportunities may be anticipated or unanticipated prior to the lesson. A teacher who has experience of teaching mathematics to this age group and who is familiar with the mathematics curriculum and with children's thinking is more likely to be able to anticipate such opportunities and to accommodate them in one's goals, orientations and knowledge resources. I now consider each of these aspects in more detail.

Teacher Goals

When studying the transcript for the lesson excerpt, at least ten implicit goals can be identified. Most goals were mathematical but two related to classroom management and establishing classroom norms. See Figure 1.

Figure 1. Mathematical goals identified in lesson excerpt 1.

<p>Mathematical</p> <ul style="list-style-type: none">A. Have students evaluate number sentences as true or false and justify their judgmentB. Highlight a student's contribution that is related to a key lesson goal or that is central to mathematics more generallyC. Seek comment and responses to student contributions from other studentsD. Ask a student to reword contribution of another studentE. Identify connections between children's methodsF. Elicit a description of a procedure usedG. Respond to a student's commentH. Provide support to a student who is stuck <p>Classroom Management & Establishing Classroom Norms</p> <ul style="list-style-type: none">I. Ensure that teacher and all students hear each student who addresses the classJ. Affirm constructive classroom behaviour
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The teacher's primary goal throughout the clip was to have students decide if the number sentences were true or false and to justify their judgments. On some occasions more than one justification was sought and given by students. For example, in the sentence $58 + 76 = 354$ one student said that it was false "because I did the sum." Another pupil, Jack, agreed that it was false but gave a different reason which involved estimation rather than calculating the sum:

Because like five and seven is twelve and you couldn't get three hundred. Like fifty and seventy are one hundred and twenty and eight and six wouldn't get you to three hundred and fifty four so you just know it's wrong straight away by looking at it.

In this statement the student indicates how he had estimated the sum of the numbers (by adding the number of tens in each number) and although this required some unpacking, he followed up by pointing out that the sum did not need to be calculated because it was so far from the sum stated in the number sentence. Jack has made a useful step here by showing that the veracity of a number sentence can be established without completing the operation in the sentence.

An example of the teacher pursuing a classroom management goal occurred when a question was asked and students quietly raised their hands to offer the answer. The teacher made the following comment:

What's really good again is how you are giving some people a chance just to figure it out, just to take their time and to think it through and you're waiting patiently. That's a really good thing that I like that about this class, you're doing really well at this.

At the start of a new year with new students a teacher needs to establish classroom norms that will benefit the work of the class during the year. By affirming behaviour which allowed other students to think through their answer, the teacher hoped to encourage such behaviour throughout their time working together.

Teacher Orientations

In the excerpt teacher orientations toward learning, teaching, students, classroom environments and towards mathematics were evident. A total of thirteen orientations were identified and these are listed in Figure 2.

Figure 2. Teacher orientations identified in lesson excerpt 1.

<p>ORIENTATIONS</p> <p>Toward learning</p> <ul style="list-style-type: none">L1. Students can learn from one another and can use one another's ideasL2. Students can learn from hearing others describe procedures usedL3. Students learn from paraphrasing other students' explanations <p>Toward teaching</p> <ul style="list-style-type: none">T1. Students' ideas need to be probedT2. Some ideas from students need to be highlightedT3. Some ideas need to be challenged <p>Toward Students</p> <ul style="list-style-type: none">S1. All have important ideas to contribute <p>Toward Classroom Environments</p> <ul style="list-style-type: none">C1. All students need to be able to hear one another and all students are worth hearingC2. Students need to give one another time to think, even if they know the answer <p>Toward Mathematics</p> <ul style="list-style-type: none">M1. Statements, whether true or false, require explanation or justificationM2. Some ideas are worth pursuing even if they are not part of the focus of the lessonM3. More than one explanation or justification of a response is possibleM4. A problem can be solved by referring to a problem already solved
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Throughout the excerpt the teacher asked pupils to justify statements they made, whether the statements were true or false. On one occasion a student who was asked to justify a wrong answer

corrected the answer without any prompting. On occasion it was necessary to ask a student to complete a justification that was incomplete. For example, when Dion claimed that $345 + 568 - 568 = 353$ was false the following exchange took place:

Teacher: False. Why is it false?

Dion: Because three hundred and forty five doesn't equal to three hundred and fifty three.

Teacher: But you're forgetting about these two numbers in here [pointing to $568 - 568$]

Dion: Yeah, three hundred and forty five plus five hundred and sixty eight minus five hundred and sixty eight equals three hundred and forty five but that has down three hundred and fifty three.

In justifying his claim that the sentence was false, Dion abbreviated his justification. The teacher pushed for an elaboration of the justification which Dion provided.

As previously alluded to, another orientation held by the teacher was that some mathematical ideas are worth pursuing even if they are not part of the focus of the lesson. For example, one student, Daniel, added $27 + 16$ to justify why the sentence $43 = 27 + 16$ is true. However, the teacher pursued the following line of questioning:

Teacher: How did you figure that out, how did you do that because you didn't do any writing or anything like that so how did you figure out in your head that twenty seven and sixteen is forty three?

Daniel: I added seven plus six and carried the one.

Teacher: Okay. I don't think Jack and Fionn can hear you. Can you say it again, how did you add the twenty seven and sixteen?

Daniel: I added seven and six in my head and took the one plus two and the one.

Teacher: Okay so I think what you're saying is you added the six and the seven and that gave you what?

Daniel: Thirteen.

Teacher: And then what did you do?

Daniel: I carried the one, so two plus two is four.

Teacher: So, you've the one here and then you've the two and then you had that other one that you carried in your head. Okay that's good.

In this excerpt the teacher asked Daniel to describe in detail the procedure he used to add the two numbers because he had done the calculation without using pencil and paper. Given that these

children were in third class and most could be assumed to be able to add 27 and 16 mentally, there was little need for the teacher to pursue this line of questioning. However, on this occasion and again later in the excerpt this was done. This is where the teacher's orientations seemed to be interacting with the overarching mathematical goal of the lesson.

Teacher Resources

The knowledge available to the teacher was his experience as a teacher and his experience as a teacher educator. He had not explicitly taught this topic before but he had taught mathematics to students and to student teachers for a total of almost twenty years. This would have given him knowledge about students of this age in general and their needs in learning mathematics. He pursued postgraduate studies in mathematics education and had access to a wide range of reading material on the subject. This would have provided him with access to a range of subject matter knowledge, including knowledge of content and knowledge of teaching methods. It would also have introduced him to sources of suitable problems to use. The material used in this excerpt was drawn from one source (Carpenter et al., 2003).

In responding to one student, the teacher would have benefited from anticipating a problem the student had. The sentence was $345 + 568 - 568 = 353$. The student who was called on to answer stated that the sentence was true. It transpired that the student had not understood the previous, (similar and easier) problem which was solved by knowing that $48 - 48 = 0$. The student eventually stated that "forty-eight take away forty-eight" equals zero and the teacher wanted to check if the student understood the general idea that $x - x = 0$. He posed several calculations for the student along the lines of "one hundred and twenty-six take away one hundred and twenty-six." A better test of the student's understanding would have been to ask her to compose her own addition and subtraction calculations that equalled zero because this would have required her to devise number sentences of that form. However, that was not done in this interaction.

On the other hand, the teacher demonstrated knowledge of children learning this topic when, in an example mentioned above, a student justified claiming that $58 + 76 = 354$ was false because she "did the sum." The teacher looked for agreement that it was false and in the process elicited the important statement that "you just know it's wrong straight away by looking at it." This view was reiterated by another student who stated that "if you were going to get three hundred you'd have to be adding higher numbers." Eliciting this idea for this number sentence was important because this knowledge

encourages students to look at number sentences in their entirety rather than as individual numbers that have to be operated on. This view of number sentences is an important step in learning algebra.

Another example of the teacher drawing on his mathematical knowledge occurred after a student had justified a claim that $12 - 9 = 3$ is true “by doing twelve minus nine ... is three.” A second student, Mona, introduced “another way you could do it,” by which she meant to ascertain “if three is the right answer.”³ Her method to “get it to be more right” was “by adding three and nine and then three and nine is twelve so then you know you’re definitely right.” Mona introduced another way to justify the truth of the number sentence while also introducing an important mathematical concept: the inverse relation between addition and subtraction. The teacher highlighted this by stating to the class that Mona had said “something important in maths” and asking another student to put what she had said “into their own words.” This strategy of revoicing (Chapin, O'Connor, & Anderson, 2003) would amplify what Mona had said and check if at least one other student had heard and understood it.

Goals, Orientations and Resources

The analysis of this excerpt illustrates how goals, orientations and resources interact in the decisions made by a teacher in the practice of teaching. In individual teaching moments one component can be seen to exert a greater or lesser impact. But to understand how planned practice is implemented, attention needs to be paid to all three.

The analysis presented here illustrates that a teacher is always trying to achieve multiple goals and which one takes precedence in any decision-making process is determined by a complex interaction among the teacher’s goals, orientations and resources. Better understanding of this interaction is important in educating pre-service and in-service teachers for the work of teaching mathematics because it can subsequently influence the opportunities given to children to develop their mathematical thinking. The analysis also indicates that even if a teacher has mathematical knowledge for teaching, activating it in a particular context requires holding the knowledge in a particular way. Rowland and his colleagues refer to this ability to “think on one’s feet” as contingency knowledge (Rowland, Huckstep, & Thwaites, 2005). Finally, the analysis of this excerpt suggests that a better understanding of the interaction among goals, orientations and resources is a worthwhile area of research which would benefit from further study.

³ Note that by using the term “answer” in this context the second student is interpreting the equals sign as an operator.

5. How the research has contributed to your professional learning?

The research has contributed to my professional learning in multiple ways. It was important for me to develop lesson plans where the focus was simultaneously focused on the mathematics content and the mathematics practice of the primary school curriculum. Too often developing the mathematics skills and practices is overlooked. The research showed what is possible when children are encouraged and given the opportunity to explain their reasoning.

11. How this research will benefit the teaching profession and the wider education community

This work is significant for mathematics education scholars for four reasons: the topic itself, the method of analysis, its relevance for research-based practice in mathematics education and as a case of how mathematical content and practices can be integrated. First, little has been documented or analysed about primary school teaching of equivalence as a relation that includes more than the equality relation. This study analyses one instance of teaching the topic with a wider goal.

Second, although much mathematics education research has been produced about teacher knowledge, or teacher beliefs on their own, this study investigates how knowledge, beliefs and goals interact in context. In doing this, the study acknowledges the complexity of coordinating instruction and it uses a model that has been used previously to analyse mathematics teaching (e.g. Schoenfeld, 1998).

Third, the study provides data for researchers or practitioners who wish to design or implement a teaching unit on equivalence that is well-documented and that itself is informed by research on the topic. Not only does the report identify opportunities and challenges that were encountered in implementing this unit, but it uses rich and varied data to justify claims about the opportunities and challenges.

Finally, the paper shows how a teacher can integrate teaching mathematics content and mathematics practices in teaching. The children are learning about equivalence, but they are also learning how to reason and justify their mathematical ideas.

The material collected as part of this research project has already been of benefit to the teaching profession and the wider education community. Video extracts from the lessons and samples of children's work have been used in presentations I have given to teachers, principals and inspectors over the last decade. Such presentations include:

Practitioner Audience

- Irish Primary Principals' Conference (2012). "Principals Leading Numeracy" (<http://seandelaney.com/wp-content/uploads/2012/01/Delaney-IPPN-2012-Delivered.pdf>) (Approximately 200 principals in attendance and presentation posted online).

Policy-Maker Audience

- Presentation to Primary School Inspectors on 6 September (2014), "What to Consider When Evaluating Mathematics Teaching" (Approximately 80 inspectors in attendance).

Researcher Audience

- (American Educational Research Association) AERA Conference (2011) in New Orleans. "Teaching in Context: Equivalence in the Elementary School (Approximately 20 researchers in attendance)."
- Educational Studies Association of Ireland (ESAI) Conference on 26 March (2010). "Teaching Equivalence to Promote Children's Mathematical Thinking: A Study of Practice (Approximately 20 researchers in attendance).
- (Standing Conference on Teacher Education North and South) SCOTENS Conference in Cavan on 29 September (2011). Learning to Teach Primary School Mathematics. (<http://seandelaney.com/wp-content/uploads/2011/09/Learning-to-Teach-Primary-School-Math-SCoTENS-30-9-11.pdf>) (Approximately 20 researchers in attendance).

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